UNIT-I

Fourier Series

pbms

1) $(0, 2\pi)$
2) $(-\pi, \pi)$
3) Half range sine series $(0, \pi)$
4) Half range cosine series $(0, \pi)$
5) $(0, 2l)$
6) $(-l, l)$
7) Half range sine series $(0, l)$
8) Half range cosine series $(0, l)$
9) complex form of Fourier Series.
10) parseval's theorem
11) Harmonic Analysis.

UNIT-I

Fourier series

Periodic functions:-

A function $f(x)$ is said to have a period T, if for all x, $f(x+T)=f(x)$, where T is a +ve constant. The least value of $T>0$ is called the period of $f(x)$.

Example:

Let $f(x) = \sin x$

$\sin x = \sin(x+2\pi) = \sin(x+4\pi) = \cdots$

$\therefore$ The function has periods $2\pi, 4\pi, 6\pi, \cdots$

Hence 2π is the least value ~~of~~ and 2π is the period of $f(x)$.

Similarly $\cos x$ is a periodic function with the period 2π.

Note:

(i) The period of $\sin nx$ and $\cos nx$ is $\frac{2\pi}{n}$.

(ii) The period of $\sin 2x$ is π, the period of $\sin 3x$ is $\frac{2\pi}{3}$.

Dirichlet's Condition:-

A function $f(x)$ can be expanded as a fourier series in $c \le x \le c+2\pi$ if

(i) $f(x)$ is single valued and finite in $(c, c+2\pi)$

(ii) $f(x)$ has finite no. of finite discontinuities and no infinite discontinuity in $(c, c+2\pi)$

(iii) $f(x)$ has finite number of maxima ~~and~~ or minima in $(c, c+2\pi)$

Fourier Series:-

If $f(x)$ is a periodic function and satisfies Dirichlet's conditions then it can be represented by an infinite series called fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \rightarrow (1)$$

where $a_0, a_n \& b_n$ are called fourier coefficients (or) fourier constants.

where
$$a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx\,dx$$

The values of a_0, a_n and b_n are known as Euler's formula.

~~Note~~: Important Results:-

1. $\sin n\pi = 0$ 2. $\cos n\pi = (-1)^n$ 3.

Note:

The above series (1) converges to

$f(x)$ if x is a point of continuity

$\frac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity.

Important Results:-

1. $\sin n\pi = 0$ 2. $\cos n\pi = (-1)^n$ if n is an integer

3. $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

4. ~~sin~~ $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

5. $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

6. $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

7. $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

(Bernoulli formula)

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Type - I $(0, 2\pi)$

The fourier Series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

Type - I $(0, 2\pi)$

① Expand $f(x) = x^2$, $0 < x < 2\pi$ in a fourier Series if the period is 2π.

Soln:
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]$$

where
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{8\pi^3}{3}\right] = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx = \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 + 2(2\pi)\frac{\cos n(2\pi)}{n^2} - 0 - 0\right]$$

$$= \frac{1}{\pi}\left[\frac{4\pi}{n^2}(1)\right]$$

$$a_n = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-4\pi^2\left(\frac{\cos 2n\pi}{n}\right) - 0 + 0 + 2\left(\frac{\cos 2n\pi}{n^3}\right) - \frac{2\cos(0)}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2}{n} + \cancel{\frac{2}{n^3}} - \cancel{\frac{2}{n^3}}\right]$$

$\boxed{(\because (-1)^n = \cos n\pi}$

$$b_n = -\frac{4\pi}{n}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]$$

$$f(x) = \frac{8\pi^2/3}{2} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right]$$

$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx - \sum_{n=1}^{\infty}\frac{4\pi}{n}\sin nx$$

2) obtain the f...

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③ Expand in Fourier series of $f(x) = x \sin x$ for $0 < x < 2\pi$ and deduce the result

$$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots \infty = \frac{\pi - 2}{4}.$$

Soln:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin x\,dx$$

$$= \frac{1}{\pi}\Big[x(-\cos x) - 1(-\sin x)\Big]_0^{2\pi}$$

$$= \frac{1}{\pi}\Big[-2\pi \cos 2\pi + 0 + 0\Big]$$

$$= \frac{1}{\pi}\Big[-2\pi(1)\Big] = -2 \Rightarrow \boxed{a_0 = -2}$$

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$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{x\,2\,\cancel{\sin x}\sin x\,\sin nx\,dx}{2}$$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin(n+1)x\,dx - \frac{1}{2\pi}\int_0^{2\pi} x\sin(n-1)x\,dx$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos(n+1)x}{(n+1)}\right] - 1\left[-\frac{\sin(n+1)x}{(n+1)^2}\right]_0^{2\pi}$$

$$- \frac{1}{2\pi}\left[x\left(-\frac{\cos(n-1)x}{(n-1)}\right) - 1\left(-\frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-2\pi\cos(n+1)2\pi}{n+1} + 0 + 0\right] - \frac{1}{2\pi}\left[-\frac{2\pi\cos(n-1)2\pi}{n-1} + 0 + 0\right]$$

$$= \frac{1}{\cancel{2\pi}}\left[\frac{-\cancel{2\pi}}{n+1}\right] - \frac{1}{\cancel{2\pi}}\left[\frac{-\cancel{2\pi}}{n-1}\right]$$

$$= -\frac{1}{n+1} + \frac{1}{n-1}$$

$$= \frac{-\cancel{n}+1+\cancel{n}+1}{n^2-1^2} = \frac{2}{n^2-1}$$

$$\boxed{a_n = \frac{2}{n^2-1}}, \quad (n \neq 1)$$

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$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{x\sin 2x}{2}\,dx$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{2\pi}{2}\cos 4\pi + 0 + 0\right]$$

$$= \frac{1}{2\pi}\left[-\pi(1)\right]$$

$$\boxed{a_1 = -\frac{1}{2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin nx\sin x\,dx$$

$\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$$= \frac{1}{\pi}\int_0^{2\pi} x\,\frac{1}{2}\left[\cos(n-1)x - \cos(n+1)x\right]dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\cos(n-1)x\,dx - \frac{1}{2\pi}\int_0^{2\pi} x\cos(n+1)x\,dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{\sin(n-1)x}{n-1}\right) - 1\left(-\frac{\cos(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}$$

$$- \frac{1}{2\pi}\left[x\left(\frac{\sin(n+1)x}{n+1}\right) - 1\left(\frac{-\cos(n-1)x}{(n+1)^2}\right)\right]^{2\pi}$$

$$= \frac{1}{2\pi}\left[0 + \frac{\cos(n-1)2\pi}{(n-1)^2} - \frac{\cos(n-1)(0)}{(n-1)^2}\right]$$

$$- \frac{1}{2\pi}\left[0 + \frac{\cos(n+1)2\pi}{(n+1)^2} - \frac{\cos 0}{(n+1)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n-1)^2}\right] - \frac{1}{2\pi}\left[\frac{1}{(n+1)^2} - \frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{2\pi}(0) - \frac{1}{2\pi}(0)$$

$\boxed{b_n = 0}$ $(n \neq 1)$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x \, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin^2 x \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(x - x\cos 2x)\, dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} - \frac{1}{2\pi}\left[x\left(\frac{\sin 2x}{2}\right) + \frac{\cos 2x}{4}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{4\pi^2}{2}\right] - \frac{1}{2\pi}\left[0 + \frac{\cos 4\pi}{4} - \frac{\cos 0}{4}\right]$$

$$= \pi - \frac{1}{2\pi}\left[\frac{1}{4} - \frac{1}{4}\right]$$

$$= \pi - 0$$

$\boxed{b_1 = \pi}$

The fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$f(x) = \frac{-2}{2} + \left(\frac{-1}{2}\cos x\right) + \sum_{n=2}^{\infty} \frac{2}{n^2-1} \cos nx + \pi \sin x + \sum_{n=2}^{\infty} (0) \sin nx$$

$$x \sin x = -1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty} \frac{\cos nx}{n^2-1} + \pi \sin x \longrightarrow (1)$$

put $x = \pi/2$

$$f(\pi/2) = \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{\pi}{2}(1) = \pi/2$$

From (1) $\Rightarrow$

$$\frac{\pi}{2} = -1 - \frac{1}{2}\cos\frac{\pi}{2} + 2\sum_{n=2}^{\infty}\left(\frac{1}{n^2-1}\right)\cos\frac{n\pi}{2} + \pi \sin\frac{\pi}{2}$$

$$\frac{\pi}{2} = -1 - \frac{1}{2}(0) + 2\sum_{n=2}^{\infty} \frac{1}{n^2-1}\cos\frac{n\pi}{2} + \pi$$

$$\frac{\pi}{2} - \pi + 1 = 2\sum_{n=2}^{\infty} \frac{1}{n^2-1}\cos\frac{n\pi}{2}$$

$$1 - \pi/2 = 2\sum_{n=2}^{\infty} \frac{1}{n^2-1}\cos n\pi/2$$

$$\frac{2-\pi}{2} = 2\sum_{n=2}^{\infty} \frac{1}{n^2-1}\cos\frac{n\pi}{2}$$

$\cos\frac{3\pi}{2} = \cos(\pi + \pi/2) = \cos\pi/2 = 0$

$\cos\frac{5\pi}{2} = \cos(2\pi + \pi/2) = \cos\pi/2$

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}\cos\frac{n\pi}{2} = \frac{2-\pi}{4}$$

$$\sum_{n=2}^{\infty} \frac{1}{(n-1)(n+1)}\cos n\pi/2 = \frac{2-\pi}{4}.$$

$$\frac{1}{1\cdot 3}(-1) + \frac{1}{2\cdot 4}(0) + \frac{1}{3\cdot 5}(1) + \cdots = \frac{2-\pi}{4}.$$

$$-\left[\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots\right] = \frac{2-\pi}{4}$$

$$\frac{1}{} - \frac{1}{} + \frac{1}{} - \cdots = \pi - 2/4$$

Find the F.S expansion of $f(x)=\begin{cases} x & 0\le x\le\pi \\ 2\pi-x & \pi\le x\le 2\pi\end{cases}$. deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty=\frac{\pi^2}{8}$

change of interval

interval $(0, 2l)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{l}\right)+b_n\sin\left(\frac{n\pi x}{l}\right)\right]$$

where $a_0=\frac{1}{l}\int_0^{2l} f(x)\,dx$

$$a_n=\frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n=\frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1,3,5}^{\infty}\frac{\cos nx}{n^2}$$

$x=0$

$$f(0)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow -\frac{\pi}{2}=-\frac{4}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\Rightarrow\frac{\pi^2}{8}=\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}$$

Interval $(c, c+2\ell)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{\ell}\right)\right] + \sum_{n=1}^{\infty}\left[b_n \sin\left(\frac{n\pi x}{\ell}\right)\right]$$

where $a_0 = \frac{1}{\ell}\int_{c}^{c+2\ell} f(x)\,dx.$

$$a_n = \frac{1}{\ell}\int_{c}^{c+2\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right) dx.$$

$$b_n = \frac{1}{\ell}\int_{c}^{c+2\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$$

Interval $(-\ell, \ell)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{\ell}\right)\right] + \sum_{n=1}^{\infty}\left[b_n \sin\left(\frac{n\pi x}{\ell}\right)\right]$$

$$a_0 = \frac{1}{\ell}\int^{\ell} f(x)\,dx$$

Soln:

Soln:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{l}\left[\int_0^{l} (l-x)\,dx + \int_l^{2l} (0)\,dx\right]$$

$$\left(\frac{(l-x)^2}{-2}\right)_0^l$$

$$0 + \frac{l^2}{2}$$

$$= \frac{1}{l}\left[lx - \frac{x^2}{2}\right]_0^l$$

$$= \frac{1}{l}\left[l^2 - \frac{l^2}{2}\right] = \frac{1}{l}\left[\frac{l^2}{2}\right]$$

$$\boxed{a_0 = \frac{l}{2}}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\int_0^{l} (l-x)\cos\frac{n\pi x}{l}\,dx + \int_l^{2l} (0)\,dx\right]$$

$$= \frac{1}{l}\left[\int_0^{l} (l-x)\,\frac{\cos n\pi x}{l}\,dx\right]$$

$$= \frac{1}{l}\left[(l-x)\left[\frac{\sin n\pi x/l}{n\pi/l}\right] - (-1)\left[-\frac{\cos n\pi x/l}{(n\pi/l)^2}\right]\right]_0^l$$

$$= \frac{1}{l}\left[0 - \frac{\cos\frac{n\pi l}{l}}{\frac{n^2\pi^2}{l^2}} + \frac{\cos 0}{\frac{n^2\pi^2}{l^2}}\right]$$

$$= \frac{1}{l}\left[\frac{-(-1)^n}{\frac{n^2\pi^2}{l^2}} + \frac{1}{n^2\pi^2/l^2}\right]$$

$$a_n = \frac{1}{l}\cdot\frac{l^2}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$a_n = \frac{1}{n^2\pi^2}\left[1-(-1)^n\right] = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2l}{n^2\pi^2} & \text{if } n \text{ is odd.} \end{cases}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left[\int_0^{l}(l-x)\sin\left(\frac{n\pi x}{l}\right) + 0\right]$$

$$= \frac{1}{l}\left[(l-x)\left(-\frac{\cos(n\pi x/l)}{n\pi/l}\right) - (-1)\left(-\frac{\sin(n\pi x/l)}{n^2\pi^2/l^2}\right)\right]_0^{l}$$

$$= \frac{1}{l}\left[0 + l\frac{\cos(0)}{n\pi/l} - \frac{\sin n\pi l/l}{n^2\pi^2/l^2} + 0\right] \qquad (\sin n\pi = 0)$$

$$= \frac{1}{l}\left[\ \ l^2 \ \ \right]$$

$$0 = \frac{l}{4} + \frac{2l}{\pi^2} \sum_{n=odd}^{\infty} \frac{(-1)^n}{n^2}$$

$$\frac{-l}{4} = \frac{2l}{\pi^2} \sum_{n=odd}^{\infty} \frac{(-1)^n}{n^2}$$

$$\frac{2}{\pi^2}\left[\frac{-1}{1^2} - \frac{1}{3^2} - \frac{1}{5^2} - \cdots \right] = -\frac{1}{4}.$$

$$\boxed{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty = \frac{\pi^2}{8}}$$

put $x = l/2$

$$l - \frac{l}{2} = \frac{l}{4} + \frac{2l}{\pi^2} \sum_{n=odd}^{\infty} \frac{\cos\left(\frac{n\pi l/2}{l}\right)}{n^2} + \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi l/2}{l}\right)$$

$$\frac{l}{2} = \frac{l}{4} + \frac{2l}{\pi^2}(0) + \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) \quad (\cos 90^\circ = 0)$$

$$\frac{l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) = \frac{l}{2} - \frac{l}{4}$$

$$\frac{l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) = \frac{l}{4}$$

$\sin 3\pi/2 = (\pi + \pi/2)$
$= -\sin \pi/2$
$= -1$

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) = \pi/4$$

$$\boxed{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \pi/4}$$

② Find a fourier series to represent $f(x) = 2x - x^2$ with period 3 in the range $(0, 3)$

$(0, 3)$

Soln: Here $2l = 3$
$\Rightarrow l = 3/2$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$a_0 = \frac{1}{(3/2)}\int_0^3 (2x - x^2)\,dx$$

$$= \frac{1}{(3/2)}\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]$$

$$= \frac{2}{3}\left[9 - \frac{27}{3}\right] = \frac{2}{3}\left[\frac{27-27}{3}\right] = \frac{2}{3}(0)$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx.$$

$$= \frac{1}{(3/2)}\int_0^3 (2x - x^2)\cos\frac{n\pi x}{(3/2)}\,dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[(2x - x^2)\left[\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\frac{2n\pi}{3}}\right] - (2-2x)\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\frac{4n^2\pi^2}{9}} + (-2)\frac{-\sin\left(\frac{2n\pi x}{3}\right)}{\frac{8n^3\pi^3}{27}}\right]_0^3$$

$$= \frac{2}{3}\left[0 + (-4)\frac{\cos 2n\pi}{\frac{4n^2\pi^2}{9}} - 2\frac{\cos 0}{\frac{4n^2\pi^2}{9}} + 0\right]$$

$$= \frac{2}{3}\left[\frac{-\cancel{36}^{\,9}}{\cancel{4}n^2\pi^2}(1) - \frac{\cancel{2}\times 9}{\cancel{4}_{\,2}n^2\pi^2}\right]$$

$$= \frac{2}{3}\left[\frac{-9}{n^2\pi^2} - \frac{9}{2n^2\pi^2}\right]$$

$$a_n = \frac{\cancel{2}}{3}\left[\frac{-18-9}{\cancel{2}n^2\pi^2}\right] = \frac{1}{\cancel{3}}\left[\frac{-\cancel{27}^{\,9}}{n^2\pi^2}\right] = \frac{-9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{3}\int_0^{3} (2x - x^2)\sin\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[(2x-x^2)\left(\frac{-\cos\frac{2n\pi x}{3}}{2n\pi/3}\right) - (2-2x)\left(\frac{-\sin 2n\pi/3}{4n^2\pi^2/9}\right) + (-2)\left(\frac{\cos 2n\pi/3}{2^3n^3\pi^3/3^3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[(-3)\left(\frac{-\cos 2n\pi}{(2n\pi/3)}\right) + 0 + 0 - 2\,\frac{\cos 2n\pi}{\frac{2^3n^3\pi^3}{3^3}} + 2\,\frac{\cos 0}{\frac{2^3n^3\pi^3}{3^3}}\right]$$

$$= \frac{2}{3}\left[\frac{-9}{2n\pi}(-1) - \frac{2\times 3^3}{2^3\times n^3\pi^3}(1) + \frac{2\times 3^3}{2^3n^3\pi^3}\right]$$

$$= \frac{2}{3}\left[\frac{9}{2n\pi} - \frac{27}{4n^3\pi^3} + \frac{27}{4n^3\pi^3}\right]$$

$$b_n = \frac{2}{3}\left[\frac{9}{2n\pi}\right] = \frac{3}{n\pi}$$

$$\boxed{b_n = \frac{3}{n\pi}}$$

$$\therefore\ f(x) = \frac{-9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{3}\right)$$

period $2l$ for the

Type – II $(-\pi < x < \pi)$

① Find the fourier series of $f(x) = x^2 + x$ in $(-\pi, \pi)$ of periodicity 2π Hence deduce for $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Soln:

neither odd nor even

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x)\,dx$$

$$= \frac{1}{\pi}\Big[\underbrace{\int_{-\pi}^{\pi} x^2 dx}_{\text{(even)}} + \underbrace{\int_{-\pi}^{\pi} x\,dx}_{\text{(odd)}}\Big]$$

$$= \frac{1}{\pi}\Big[2\int_0^{\pi} x^2 dx + 0\Big]$$

$$= \frac{2}{\pi}\Big[\frac{x^3}{3}\Big]_0^{\pi} = \frac{2}{\pi}\Big[\frac{\pi^3}{3}\Big]$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x)\cos nx\,dx$$

$$= \frac{1}{\pi}\Big[\underbrace{\int_{-\pi}^{\pi} x^2\cos nx\,dx}_{\text{(Even)}} + \underbrace{\int_{-\pi}^{\pi} x\cos nx\,dx}_{\text{(odd)}}\Big]$$

$$= \frac{1}{\pi}\Big[2\int_0^{\pi} x^2\cos nx\,dx + 0\Big]$$

$$= \frac{2}{\pi}\Big[x^2\Big(\frac{\sin nx}{n}\Big) - 2x\Big(\frac{-\cos nx}{n^2}\Big) + 2\Big(\frac{-\sin nx}{n^3}\Big)\Big]_0^{\pi}$$

$$= \frac{2}{\pi}\Big[0 + \frac{2\pi\cos n\pi}{n^2} - 0 + 0\Big]$$

$$= \frac{2}{\pi}\Big[\frac{2\pi}{n^2}(-1)^n\Big]$$

$$a_n = \frac{4}{n^2}(-1)^n. \qquad [\cos n\pi = (-1)^n]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x^2+x)\sin nx\,dx$$

$$= \frac{1}{\pi}\Big[\underbrace{\int_{-\pi}^{\pi} x^2 \sin nx\,dx}_{\text{(odd)}} + \underbrace{\int_{-\pi}^{\pi} x\sin nx\,dx}_{\text{(even)}}\Big]$$

$$= \frac{1}{\pi}\left[0 + 2\int_0^{\pi} x\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n} + 0 + 0\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{n}(-1)^n\right]$$

$$b_n = -\frac{2}{n}(-1)^n.$$

The fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos nx + b_n \sin nx\right]$$

$$f(x) = \frac{\frac{2\pi^2}{3}}{2} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}(-1)^n\cos nx\right] + \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n \sin nx$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}(-1)^n\cos nx - \frac{2}{n}(-1)^n\sin nx\right]$$

put $x=\pi$

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos n\pi - 0$$

$$\Rightarrow \pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n(-1)^n$$

$$\frac{2\pi^2}{4\cdot 3} = \sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}}$$

$x = -\pi$

$f(-\pi) = \pi^2 - \pi$

$f(\pi) = \pi^2 + \pi$

$$\frac{f(-\pi)+f(\pi)}{2} = \frac{2\pi^2}{2} = \pi^2$$

Even function:-

If $f(-x) = f(x)$ then $f(x)$ is even function.

odd function:

If $f(-x) = -f(x)$ then $f(x)$ is odd function.

$\int_{-\pi}^{\pi} f(x)dx = 0$, $f(x)$ is odd.

$\int_{-\pi}^{\pi} f(x)dx = 2\int_{0}^{\pi} f(x)dx$, $f(x)$ is even

---

Type - III

Even and odd functions in $(-\pi, \pi)$

Even function:-

If $f(x)$ is an even function then,

the fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$, $a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$

and $b_n = 0$.

odd function:-

If $f(x)$ is an odd function then,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

where $a_0 = 0$, $a_n = 0$ & $b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$.

Note: Even and odd function cases can be used only when the function $f(x)$ is defined in $(-\pi,\pi)$ (or) $(-l,l)$.

* Even fun × even fun = Even fun
* odd fun × Even fun = odd fun
* Even fun × odd fun = odd fun.
* odd fun × odd fun = even fun.

② Find b_n in the expansion of x^2 as a fourier series in $(-\pi, \pi)$.

Soln: Given $f(x) = x^2$ is an even function in $(-\pi, \pi)$

$\therefore b_n = 0$.

③ If $f(x)$ is an odd function defined in $(-l, l)$ what are the values of a_0 and a_n?

Soln: Given $f(x)$ is an odd function in $(-l, l)$

$\therefore a_0 = 0, a_n = 0$.

④ Find the fourier constants b_n for $x \sin x$ in $(-\pi, \pi)$

Soln:

$f(x) = x \sin x$

$f(-x) = -x \sin(-x)$

$f(-x) = x \sin x$

$f(-x) = f(x)$

$\therefore f(x)$ is an even function

Hence $b_n = 0$.

(8) Find the fourier series of $f(x) = \cos ax$ in $-l \le x \le l$. where 'a' is not an integer.

<u>Soln:</u> Here $f(x) = \cos ax$ is an even function.

The fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l \cos ax\,dx$$

$$= \frac{2}{l}\left[\frac{\sin ax}{a}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{\sin al}{a} - 0\right]$$

$$a_0 = \frac{2}{al}\sin al$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l \cos ax \cos\left(\frac{n\pi x}{l}\right)dx$$

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$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$\cos A\cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

$$= \frac{1}{l}\int_0^l \left[\cos\left(\frac{n\pi}{l}+a\right)x + \cos\left(\frac{n\pi}{l}-a\right)x\right]dx$$

$$= \frac{1}{l}\left[\frac{\sin\left(\frac{n\pi}{l}+a\right)x}{\left(\frac{n\pi}{l}+a\right)} + \frac{\sin\left(\frac{n\pi}{l}-a\right)x}{\left(\frac{n\pi}{l}-a\right)}\right]_0^l$$

$$= \frac{1}{l}\left[l\left(\frac{\sin\left(\frac{n\pi}{l}+a\right)l}{n\pi+al} + \frac{\sin\left(\frac{n\pi}{l}-a\right)l}{n\pi-al}\right)\right] - 0$$

$$= \frac{\sin(n\pi+al)}{n\pi+al} + \frac{\sin(n\pi-al)}{n\pi-al}$$

$$= \frac{(-1)^n\sin al}{n\pi+al} - \frac{(-1)^n\sin al}{n\pi-al}$$

$\sin(n\pi \pm al) = \sin n\pi\cos al \pm \cos n\pi\sin al$
$= \pm(-1)^n\sin al$

$$= (-1)^n\sin al\left[\frac{n\pi-al-n\pi-al}{(n\pi)^2-(al)^2}\right]$$

$$a_n = \frac{-2al}{n^2\pi^2-a^2l^2}(-1)^n\sin al$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{\frac{2\sin al}{al}}{2} + \sum_{n=1}^{\infty}\left(\frac{-2al}{n^2\pi^2-a^2l^2}\right)(-1)^n\sin al\cos\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{\sin al}{al} - 2al\sin al\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\pi^2-a^2l^2}\cos\left(\frac{n\pi x}{l}\right)$$

(10) Find a Fourier Series to represent $f(x) = \pi x$ in the interval $-l < x < l$.

Soln:

$f(x) = \pi x$ is an odd function.

$f(x) = \pi x$

$f(-x) = \pi(-x) = -\pi x$

$f(-x) = -f(x)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{2}{l}\int_0^l \pi x \sin\left(\frac{n\pi x}{l}\right)dx.$$

$$= \frac{2\pi}{l}\left[x\left[\frac{-\cos\left(\frac{n\pi x}{l}\right)}{(n\pi/l)}\right] - 1\left[\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\frac{n^2\pi^2}{l^2}}\right]\right]_0^l$$

$$= \frac{2\pi}{l}\left[\frac{-l\cos\left(\frac{n\pi l}{l}\right)}{n\pi/l} + 0 + 0\right]$$

$$= \frac{2\pi}{l}\left[-l \times \frac{l}{n\pi}\cos n\pi\right]$$

$$b_n = -\frac{2l}{n}(-1)^n$$

$$f(x) = \sum_{n=1}^{\infty} -\frac{2l}{n}(-1)^n \sin\left(\frac{n\pi x}{l}\right)$$

$$f(x) = -2l\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{l}\right)$$

(3) Find F.S expansion of the periodic function $f(x)$ of period $2l$ defined by

$$f(x) = \begin{cases} l+x & -l \le x \le 0 \\ l-x & 0 \le x \le l. \end{cases}$$

deduce that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \pi^2/8$

$a_0 = l$, $a_n = \frac{4l}{n^2\pi^2}$ odd

$n = 0$

Obtain the F.S for $f(x) = -x$ when $-\pi < x < 0$ & $f(x) = x$ when $0 < x < \pi$ & deduce that $1/1^2 + 1/3^2 + 1/5^2 + \dots = \pi^2/8$.

(1) Obtain the Fourier Series to represent the function (or) Obtain the F.S $f(x) = |x|$, $-\pi < x < \pi$ and hence deduce $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

Ans: $[a_0 = \pi$, $a_n = \frac{-4}{\pi n^2}$ if n is odd $b_n = 0$, $n = 0$.

$$\sum_{n=odd}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

(2) Find the Fourier expansion of $f(x) = x$ in the interval $(-\pi, \pi)$

H.W

Half Range Series $(0, \pi)$.

Half range sine series: -

The half-range sine series of $f(x)$ defined in the interval $0<x<\pi$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx .$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$.

Half range Cosine series: -

The half range cosine series of $f(x)$ defined in the interval $0<x<\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx .$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx .$$

Sine series: -

In $0<x<l$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $b_n = \frac{2}{l}\int_0^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

Cosine series In $0<x<l$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$ & $a_n = \frac{2}{l}\int_0^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$

① Find the half range fourier sine series for

$f(x) = x$ in $0 < x < \pi$.

Soln: Given $f(x) = x$ in $0 < x < \pi$

The fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx.$$

$$= \frac{2}{\pi} \left[x\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[-\frac{\pi \cos n\pi}{n} + 0 + 0 \right]$$

$$= \frac{2}{\pi} \left[-\frac{\pi}{n} (-1)^n \right]$$

$$b_n = -\frac{2}{n} (-1)^n.$$

The half range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \sum_{n=1}^{\infty} \left(-\frac{2}{n}\right) (-1)^n \sin nx$$

$$\boxed{x = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx}$$

② By using the sine series for unity in $0<x<\pi$

Show that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$ (parseval's identity)

Soln: Here $f(x)=1$, $0<x<\pi$

The fourier sine series for $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} 1.\sin nx\, dx$$

$$= \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\cos n\pi}{n} + \frac{\cos 0}{n}\right]$$

$$b_n = \frac{2}{\pi}\left[-\frac{(-1)^n}{n} + \frac{1}{n}\right]$$

$$b_n = \begin{cases} \frac{4}{\pi n} & \text{when } n \text{ is odd} \\ 0 & \text{when } n \text{ is even} \end{cases}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

$$= \sum_{n=1,3,5} \frac{4}{\pi n}\sin nx$$

$$= \frac{4}{\pi} . \sum_{n=1,3,5} \frac{\sin nx}{n}$$

$$= \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$$

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By parseval's identity,

$$\bar{y}^2 = \frac{1}{2}\sum_{n=1}^{\infty} b_n^2 \qquad (\because a_0=0,\ a_n=0)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{4}{\pi(2n-1)}\right]^2$$

$$= \frac{1}{2}\cdot\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$= \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

Also $\bar{y}^2 = \frac{1}{\pi-0}\int_0^{\pi}[f(x)]^2\,dx$

$$= \frac{1}{\pi}\int_0^{\pi} 1\cdot dx$$

$$= \frac{1}{\pi}[x]_0^{\pi}$$

$$= \frac{1}{\pi}[\pi]$$

$$\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = 1$$

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}}$$

④ Find the half range cosine series of $f(x) = x(2-x)$ in $0 \le x \le 2$.

Deduce the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

Soln: $f(x) = x(2-x)$, $0 \le x \le 2$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$ be the half range cosine series.

Here $l = 2$.

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \int_0^2 [x(2-x)]\,dx$$

$$= \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2$$

$$= \left[x^2 - x^3/3\right]_0^2$$

$$= \left(4 - \frac{8}{3}\right) = \frac{12-8}{3} = \frac{4}{3}.$$

$$\boxed{a_0 = \frac{4}{3}}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{2}\int_0^2 x(2-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

effective value of $f(x)$ and is denoted by $\bar{y}$.

$$= \int_0^2 (2x - x^2)\cos\frac{n\pi x}{2}\,dx$$

$$= \left[(2x-x^2)\left(\frac{\sin\frac{n\pi x}{2}}{n\pi/2}\right) - (2-2x)\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right.$$

$$+ (2-0)\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^3\pi^3}{8}}\right)\Bigg]_0^2$$

$$= 0 + (2\cdot 4)\frac{\cos n\pi}{n^2\pi^2/4} - 0\left(\frac{1}{n^2\pi^2/4}\right)$$

$$= -8\times\frac{4}{n^2\pi^2}\cos n\pi = \frac{8\cdot 4}{n^2\pi^2}$$

$$a_n = \frac{-8}{n^2\pi^2}\left[(-1)^n + 1\right]$$

$$a_n = \begin{cases} -16/n^2\pi^2 & \text{when } n \text{ is even} \\ 0 & \text{when } n \text{ is odd} \end{cases}$$

$$\therefore f(x) = \frac{8}{6} - \frac{16}{\pi^2}\sum_{n=2,4,6}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2}$$

$$f(x) = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$$

$$f(x) = \frac{2}{3} - \frac{4}{\pi^2}\left[\frac{\cos\pi x}{1^2} + \frac{\cos 2\pi x}{2^2} + \cdots\right]$$

H.W. Find the half range cosine series for $f(x) = x^2$

① In the range $0 < x \leq \pi$ hence find the ~~sum~~ sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$ (Ans: $\frac{\pi^2}{12}$).

② Find the half-range Fourier cosine series of $f(x) = (\pi - x)^2$ in the interval $(0, \pi)$. Hence find the sum of the series $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \infty$

Ans:

$a_0 = \frac{2\pi^2}{3}$, $a_n = \frac{4}{n^2}$

$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$.

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Parseval's Theorem:—

Let $f(x)$ be a periodic function with period 2π defined in the interval $(-\pi, \pi)$ then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left[a_n^2 + b_n^2\right]$$

where a_0, a_n and b_n are fourier coefficients of $f(x)$.

Note:

$$\bar{y}^2 = \frac{1}{b-a}\int_a^b [f(x)]^2\,dx$$

R.m.s (Root mean Square)

$$\bar{y} = \sqrt{\frac{\int_a^b [f(x)]^2\,dx}{b-a}}$$

① State parseval's identity for the half range cosine expansion of $f(x)$ in $(0,1)$.

Soln: Here $b_n = 0$.

$$\frac{2}{1}\int_0^1 [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\bar{y}^2 = \frac{1}{1-0}\int_0^1 [f(x)]^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum(a_n^2)$$

$$\bar{y}^2 = \int_0^1 f(x)^2\,dx = \frac{1}{2}\left[\frac{a_0^2}{2} + \sum a_n^2\right]$$

$$\bar{y}^2 = 2\int_0^1 f(x)^2\,dx = \frac{a_0^2}{2} + \sum a_n^2$$

② Find the root mean square value of the function $f(x) = x$ in the interval $(0, l)$

Soln:

$$\text{R.m.s} = \sqrt{\frac{\int_a^b [f(x)]^2\,dx}{b-a}} \quad \text{in } (a,b)$$

$$= \sqrt{\frac{\int_0^l x^2\,dx}{l-0}} \quad a=0,\ b=l.$$

$$= \sqrt{\frac{1}{l}\left[\frac{x^3}{3}\right]_0^l}$$

$$\text{R.m.s} = \sqrt{\frac{1}{l}\left[\frac{l^3}{3}\right]} = \sqrt{\frac{l^2}{3}} = \frac{l}{\sqrt{3}} \,//$$

→ Parseval's Identity (or) Parseval's theorem:—

let $f(x)$ be a periodic function in (a,b) then

$$y^2 = \frac{1}{b-a}\int_a^b [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

note:

1) If $(a,b) = (0,\pi)$ then

i) sine series (odd function)

$$\frac{1}{\pi-0}\int_0^{\pi}[f(x)]^2 dx = \frac{1}{2}\sum_{n=1}^{\infty} b_n^2 \quad (\because a_0 \,\&\, a_n = 0)$$

(ii) cosine series (even function)

$$\frac{1}{\pi}\int_0^{\pi}[f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 \quad (\because b_n = 0)$$

2. If $(a,b) = (-\pi,\pi)$ then

$$\frac{1}{\pi-(-\pi)}\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

3. If $(a,b) = (0,2\pi)$ then

$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2).$$

4. If $(a,b) = (0,l)$ then

(i) sine series (odd function)

$$\frac{1}{l}\int_0^{l}[f(x)]^2 dx = \frac{1}{2}\sum_{n=1}^{\infty} b_n^2 \quad (\because a_0 \,\&\, a_n = 0)$$

(ii) cosine series (even function)

$$\frac{1}{l}\int_0^{l}[f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 \quad (\because b_n = 0)$$

5. If $(a,b) = (-l, l)$ then

$$\frac{1}{2l}\int_{-l}^{l}[f(x)]^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

6. If $(a,b) = (0, 2l)$ then

$$\frac{1}{2l}\int_{0}^{2l}[f(x)]^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Formulas

Interval	$f(x)$	C_n
$(-l,l)$ & $(0,2l)$	$\sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$	$\frac{1}{2l}\int_{-l}^{l} f(x) e^{-\frac{in\pi x}{l}} dx$
$(0,l)$	$\sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$	$\frac{1}{l}\int_{0}^{l} f(x) e^{-\frac{in\pi x}{l}} dx$
$(-\pi,\pi)$ & $(0,2\pi)$	$\sum_{n=-\infty}^{\infty} C_n e^{inx}$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$
$(0,\pi)$	$\sum_{n=-\infty}^{\infty} C_n e^{inx}$	$\frac{1}{\pi}\int_{0}^{\pi} f(x) e^{-inx} dx.$

① Find the complex form of the fourier series of $f(x) = e^{-x}$ in $-1 \le x \le 1$.

Soln: Given $f(x) = e^{-x}$ in $-1 \le x \le 1$.

Here $l = 1$.

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$$

where $C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-\frac{in\pi x}{l}}\, dx$

$$C_n = \frac{1}{2}\int_{-1}^{1} e^{-x} e^{-in\pi x}\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-[1+in\pi]x}\, dx$$

$$= \frac{1}{2}\left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right]_{-1}^{1}$$

$e^{in\pi} = \cos n\pi + i\sin n\pi$

$e^{in\pi} = (-1)^n + i0 = (-1)^n$

$e^{-in\pi} = \cos n\pi - i\sin n\pi$

$= (-1)^n - i0$

$e^{-in\pi} = (-1)^n$

$$= \frac{1}{2}\left[\frac{-e^{-(1+in\pi)} + e^{(1+in\pi)}}{1+in\pi}\right]$$

$$= \frac{1}{2(1+in\pi)}\left[-\left[e^{-1}\cdot e^{-in\pi}\right] + \left[e^{1}\cdot e^{in\pi}\right]\right]$$

$$= \frac{1}{2(1+in\pi)}\left[e^{1}(-1)^n - e^{-1}(-1)^n\right]$$

$$= \frac{1}{(1+in\pi)}\left[\left(\frac{e^{1}-e^{-1}}{2}\right)(-1)^n\right]$$

$$c_n = \frac{(-1)^n}{(1+in\pi)}\left[\sinh 1\right]$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(1+in\pi)} (\sinh 1)\, e^{in\pi x}$$

$$f(x) = \sinh 1 \sum_{n=-\infty}^{\infty} \frac{(-1)^n (1-in\pi)}{1+n^2\pi^2}\, e^{in\pi x}$$

② Find the complex form of the fourier series of the function $f(x) = e^x$ when $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$.

Soln: $$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where $$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x} e^{-inx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(1-in)x} dx$$

$$= \frac{1}{2\pi} \left[\frac{e^{(1-in)x}}{(1-in)} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \left[\frac{e^{(1-in)\pi} - e^{-(1-in)\pi}}{1-in} \right]$$

$$= \frac{1}{2\pi(1-in)} \left[e^{\pi} \cdot e^{-in\pi} - e^{-\pi} \cdot e^{in\pi} \right]$$

$$= \frac{1}{2\pi(1-in)} \left[e^{\pi}(-1)^n - e^{-\pi}(-1)^n \right].$$

$$= \frac{1}{\pi(1-in)} \left[\frac{e^{\pi} - e^{-\pi}}{2} \right] \cdot (-1)^n$$

$$= \frac{(-1)^n}{\pi(1-in)} \sinh\pi$$

$$c_n = \frac{(-1)^n (1+in)}{\pi(1^2+n^2)} \sinh\pi .$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (1+in)}{\pi(1^2+n^2)} \sinh\pi \cdot e^{inx}$$

$$\text{ie) } f(x) = e^{x} = \frac{\sinh\pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{1+in}{1^2+n^2} e^{inx} .$$

Harmonic Analysis

Sometimes the function is not given by a formula, but by a graph (or) by a table of corresponding values. The process of finding the Fourier series for a function given by (such values of the function and independent variable) ~~numerical value~~ is known as Harmonic Analysis.

Fourier series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$.

The fourier constants are evaluated by the following formulae:

$$a_0 = \frac{2}{n}\left[\sum f(x)\right]$$

$$a_n = \frac{2}{n}\left[\sum f(x) \cos nx\right]$$

where n – no of observations

$$b_n = \frac{2}{n}\left[\sum f(x) \sin nx\right]$$

Type-I Given Datas are in π form

Ex. ① Find the Fourier series upto the third harmonic for $y = f(x)$ in $(0, 2\pi)$ defined by the table values given below.

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	1	1.4	1.9	1.7	1.5	1.2	1.0

Soln: The length of the interval is 2π.

By omitting the value of $f(x)$ at $x = 2\pi$ we have $n = 6$.

Hence Fourier Series of $f(x)$ is given by

$$y = f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots$$
$$b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

x	y	$y\cos x$	$y\cos 2x$	$y\cos 3x$	$y\sin x$	$y\sin 2x$	$y\sin 3x$
0	1.0	1.0	1.0	1.0	0	0	0
60°	1.4	0.7	-0.7	-1.4	1.212	1.212	0
120°	1.9	-0.95	-0.95	1.9	1.645	-1.645	0
180°	1.7	-1.7	1.7	-1.7	0	0	0
240°	1.5	-0.75	-0.75	1.5	-1.299	1.299	0
300°	1.2	0.6	-0.6	-1.2	-1.039	-1.039	0
Total	8.7	-1.1	-0.3	0.1	0.519	-0.173	0

$$a_0 = \frac{2}{n}\Sigma y = \frac{2}{6}\Sigma y = \frac{1}{3}(8.7) = 2.9$$

$$a_1 = \frac{2}{n}\Sigma y\cos x = \frac{2}{6}(-1.1) = \frac{1}{3}(-1.1) = -0.366$$

$$a_2 = \frac{2}{n}\Sigma y\cos 2x = \frac{2}{6}(-0.3) = \frac{-0.3}{3} = -0.10$$

$$a_3 = \frac{2}{n}\Sigma y\cos 3x = \frac{2}{6}(0.1) = 0.033$$

$$b_1 = \frac{2}{n}\Sigma y\sin x = \frac{2}{6}(0.519) = \frac{0.519}{3} = 0.173$$

$$b_2 = \frac{2}{n}\Sigma y\sin 2x = \frac{2}{6}(-0.173) = \frac{-0.173}{3} = -0.06$$

$$b_3 = \frac{2}{6}\Sigma y\sin 3x = \frac{1}{3}(0) = 0.$$

∴ the required fourier series is

$y = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x)$,

$+ (a_3 \cos 3x + b_3 \sin 3x)$

$y = 1.45 - 0.37 \cos x + 0.17 \sin x - 0.10 \cos 2x$

$- 0.06 \sin 2x + 0.03 \cos 3x$

... two harmonic of the

Type-II Given datas are in l form:-

① Find the constant term and the coefficient of the second sine and cosine terms in the fourier expansion of y as given in the following table.

x	0	1	2	3	4	5
y	9	18	24	28	26	20.

Soln: The length of the interval is $2l = 6$

$\boxed{l = 3}$

The fourier series can be represented by

$$y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l}\right) + \left(a_2 \frac{\cos 2\pi x}{l} + b_2 \sin\frac{2\pi x}{l}\right),$$

$$y = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{3} + b_1 \frac{\sin \pi x}{3} + a_2 \cos\frac{2\pi x}{3} + b_2 \sin\frac{2\pi x}{3}$$

x	$\pi x/3$	y	$y\cos\frac{\pi x}{3}$	$y\cos\frac{2\pi x}{3}$	$y\sin\pi/3\,x$	$y\sin\frac{2\pi x}{3}$
0	0	9	9	9	0	0
1	60°	18	9	-9	15.588	15.588
2	120°	24	-12	-12	20.784	-20.784
3	180°	28	-28	28	0	0
4	240°	26	-13	-13	-22.516	22.516
5	300°	20	10	-10	-17.32	-17.32
Total		125	-25	-7	-3.464	0

$$a_0 = \frac{2}{6}\Sigma y = \frac{1}{3}(125) = 41.67$$

$$a_1 = \frac{2}{6}\Sigma y\cos\frac{\pi x}{3} = -\frac{25}{3} = -8.33$$

$$a_2 = \frac{2}{6}\Sigma y\cos\frac{2\pi x}{3} = -\frac{7}{3} = -2.33$$

$$b_1 = \frac{2}{6}\Sigma y\sin\frac{\pi x}{3} = \frac{-3.464}{3} = -1.155$$

$$b_2 = 0.$$

Q. Derive the complex form of Fourier Series for $f(x) = e^{ax}$, $-\pi < x < \pi$ given that a is a real constant. Deduce that

i) $$e^{ax} = \frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{a+in}{a^2+n^2} e^{inx}$$

ii) $$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{a \sinh a\pi}$$

Soln:-

$$f(x) = e^{ax}$$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} \longrightarrow ①$$

where

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ax} e^{-inx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(a-in)x} dx = \frac{1}{2\pi} \left[\frac{e^{(a-in)x}}{a-in} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \left[\frac{e^{(a-in)\pi}}{a-in} - \frac{e^{(a-in)(-\pi)}}{a-in} \right]$$

$e^{in\pi} = (-1)^n$

$e^{-in\pi} = (-1)^n$

$$= \frac{1}{2\pi} \left[\frac{e^{a\pi} \cdot e^{-in\pi}}{a-in} - \frac{e^{-a\pi} \cdot e^{in\pi}}{a-in} \right]$$

$$= \frac{1}{2\pi} \left[\frac{e^{a\pi} \cdot (-1)^n}{a-in} - \frac{e^{-a\pi} \cdot (-1)^n}{a-in} \right] = \frac{(-1)^n}{2\pi} \left[\frac{e^{a\pi} - e^{-a\pi}}{(a^2+n^2)} \right]$$

$$= \frac{(-1)^n}{\pi(a^2+n^2)} \left[\frac{e^{a\pi} - e^{-a\pi}}{2} \right]$$

$$C_n = \frac{(-1)^n}{\pi(a^2+n^2)} \sinh a\pi$$

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$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{inx}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\pi(a^2+n^2)} \sinh a\pi \cdot e^{inx}$$

$$f(x) = \frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{inx}}{(a^2+n^2)}$$

$x=0$

$$f(0) = \frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \cdot (1)}{a^2+n^2}$$

$$1 \times \frac{\pi}{\sinh a\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2} \; //$$

Unit-II

Fourier Transforms

fourier integral theorem (without proof) - fourier transform pair - sine and cosine Transforms - properties - Transform's of simple functions - convolution thm - parseval's identity.

Fourier integral Theorem:-

If $f(x)$ is piece-wise continuously differentiable and absolutely integrable in $(-\infty,\infty)$, then $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{is(x-t)}\, dt\, ds$.

(or) equivalently

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t) \cos\lambda(t-x)\, dt\, d\lambda.$$

This is known as fourier integral theorem (or) fourier integral formula.

Fourier Transform:- (or) Complex Fourier Transform

The Fourier Transform of $f(x)$ in $(-\infty,\infty)$ is given by

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = F(s)$$

Inverse Fourier Transform:-

The Inverse Fourier Transform of $F(s)$ is defined by

$$F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds = f(x)$$

fourier Transform and inverse fourier Transform are jointly called FTP.

Note:

i) $e^{isx} = \cos sx + i\sin sx.$

(ii) $e^{-isx} = \cos sx - i\sin sx.$

(iii) $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ if $f(x)$ is even

(iv) $\int_{-a}^{a} f(x)dx = 0$ when $f(x)$ is odd.

Convolution of two functions:- (Define Convolution)

If $f(x)$ and $g(x)$ are two functions defined in $(-\infty,\infty)$ then the convolution of $f(x)$ and $g(x)$ is defined by

$$(f*g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt.$$

Convolution Theorem:-

The Fourier Transform of the convolution of $f(x)$ and $g(x)$ is the product of their fourier transforms.

If $F[f(x)] = F(s)$ and $F[g(x)] = G(s)$ then

ie) $F[f(x) * g(x)] = F(s)\cdot G(s) = F[f(x)]\cdot F[g(x)]$

Note: $F^{-1}[F(s)\, G(s)] = f(x) * g(x) = F^{-1}[F(s)] * F^{-1}[G(s)]$

Parseval's identity:-

If $F(s)$ is the Fourier transform of $f(x)$. then $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$

Self - Reciprocal:-

If the fourier transform of $f(x)$ is $f(s)$, then $f(x)$ is said to be Self - reciprocal under fourier transform

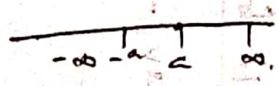
ie) $F[f(x)] = f(s)$.

$$\boxed{F(S) = f(S)}$$

Problems:-

1. Find the complex fourier transform of

$$f(x) = \begin{cases} x & \text{for } |x| \le a \\ 0 & \text{for } |x| > a \end{cases}$$



Soln: The given function can be written as

$$f(x) = \begin{cases} x & \text{if } -a \le x \le a \\ 0 & \text{if } -\infty < x < -a \text{ and } a < x < \infty. \end{cases}$$

The fourier transform of $f(x)$ is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F(S) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x(\cos sx + i \sin sx)\, dx$$

$$F(S) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \underset{(\text{odd})}{x \cos sx}\, dx + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} \underset{(\text{even})}{x \sin sx}\, dx \quad \rightarrow (1)$$

Here $x \cos sx$ is an odd function of x and $x \sin sx$ is even funtion of x.

Functions
odd × odd = even
odd × even, even × odd = odd
even × even = even

$$\therefore \int_{-a}^{a} x \cos sx\, dx = 0$$

and $\int_{-a}^{a} x \sin sx\, dx = 2\int_{0}^{a} x \sin sx\, dx$

(1) ⇒ $F(s) = 0 + \frac{i}{\sqrt{2\pi}}\, 2\int_0^a x \sin sx \, dx.$

$$F(s) = \frac{i\sqrt{2}}{\sqrt{\pi}} \int_0^a x \sin sx \, dx$$

$$= i\sqrt{\frac{2}{\pi}}\left[\left[x\left(-\frac{\cos sx}{s}\right)\right]_0^a - \int_0^a \left(-\frac{\cos sx}{s}\right)dx\right]$$

$$= i\sqrt{\frac{2}{\pi}}\left[\left(-\frac{x\cos sx}{s}\right)_0^a + \left(\frac{\sin sx}{s^2}\right)_0^a\right]$$

$$= i\sqrt{\frac{2}{\pi}}\left[-\frac{a\cos sa}{s} + 0 + \frac{\sin sa}{s^2} - 0\right]$$

$$= i\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa}{s^2} - \frac{a\cos sa}{s}\right]$$

$$= i\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa - as\cos sa}{s^2}\right]$$

$$F(s) = i\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa - sa\cos sa}{s^2}\right]$$

Use Bernoulli formula

$\int u\,dv = uv_1 - u'v_2 + u''v_3 \ldots$

(2) Find the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{in } |x| < a \\ 0 & \text{in } |x| > a \end{cases}$$

Deduce that i) $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$

(ii) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$

Soln: The given function can be written as

$$f(x) = \begin{cases} 1 & \text{in } -a < x < a \\ 0 & \text{in } -\infty < x < -a \text{ and } a < x < \infty. \end{cases}$$

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The Fourier transform is

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (1)(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \underbrace{\cos sx\, dx}_{(\text{even})} + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} \underbrace{\sin sx\, dx}_{(\text{odd})}$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{a} \cos sx\, dx + \frac{i}{\sqrt{2\pi}} (0)$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{a} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[+ \frac{\sin sx}{s} \right]_{0}^{a}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s} - \frac{\sin(0)}{s} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s} - 0 \right]$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin sa}{s} \right)$$

③ Find the Fourier transform of $f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$

Hence deduce that i) $\int_{0}^{\infty} \frac{\sin \lambda}{\lambda}\, d\lambda = \pi/2$

(ii) $\int_{0}^{\infty} \frac{\sin t}{t}\, dt = \pi/2$ (iii) $\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \pi/2$

(iv) $\int_{0}^{\infty} \frac{\sin^2 t}{t^2}\, dt = \pi/2$

Soln:

Fourier transform of f(x) is

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \qquad \left\{ \begin{array}{l} |x|<1 \\ \Rightarrow -1<x<1 \end{array} \right.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\,(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1\,(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \underset{(\text{even})}{\cos sx}\, dx + \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} \underset{(\text{odd})}{\sin sx\, dx}$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{1} \cos sx\, dx + \frac{i}{\sqrt{2\pi}}(0)$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{1} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s} \right]_{0}^{1}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin s}{s} \right]$$

Deduction:-

(i) By inversion formula

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{\sin s}{s} \right] \left[\cos sx - i \sin sx \right] ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sqrt{2}}{\sqrt{\pi}} \left[\left(\frac{\sin s}{s} \right) \cos sx - i \left(\frac{\sin s}{s} \right) \sin sx \right] ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin s}{s}\right) \cos sx \, ds - \frac{i}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s}{s}\right)\sin sx \, ds$$

even   even   even odd

$\frac{\sin s}{s}\cos sx$ is an even functions of s and

$\frac{\sin s}{s}\sin sx$ is an odd function of s.

$$\therefore f(x) = \frac{1}{\pi} 2\int_0^{\infty}\left(\frac{\sin s}{s}\right)\cos sx \, ds - \frac{i}{\pi}(0)$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\left(\frac{\sin s}{s}\right)\cos sx \, ds$$

$$\int_0^{\infty}\left(\frac{\sin s}{s}\right)\cos sx \, ds = \frac{\pi}{2} f(x) = \frac{\pi}{2}\begin{cases} 1, & |x|<1 \\ 0, & |x|>1 \end{cases} \rightarrow (1)$$

put $x=0$ in (1).

$$\int_0^{\infty}\frac{\sin s}{s}\cos s(0) \, ds = \frac{\pi}{2}$$

$$\boxed{\int_0^{\infty}\frac{\sin s}{s} \, ds = \frac{\pi}{2}}$$

Changing s to λ, we get

$$\boxed{\int_0^{\infty}\frac{\sin \lambda}{\lambda} \, d\lambda = \pi/2}$$ i) Hence proved.

(ii) changing s to t we get

$$\int_0^{\infty}\frac{\sin t}{t} \, dt = \pi/2$$

(iii) changing s to x we get

$$\int_0^{\infty}\frac{\sin x}{x} \, dx = \pi/2.$$

(iv) By parseval's identity:-

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{\sin s}{s}\right)^2 ds = \int_{-1}^{1} (1)^2 dx.$$

$$\int_{-\infty}^{\infty} \frac{2}{\pi} \left(\frac{\sin^2 s}{s^2}\right) ds = \int_{-1}^{1} dx$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2} ds = (x)_{-1}^{1} = 1+1 = 2.$$

$$\frac{\cancel{2}}{\pi} \cdot 2\int_{0}^{\infty} \frac{\sin^2 s}{s^2} ds = \cancel{2}$$

$$\frac{2}{\pi} \int_{0}^{\infty} \frac{\sin^2 s}{s^2} ds = 1$$

$$\int_{0}^{\infty} \frac{\sin^2 s}{s^2} ds = \pi/2$$

changing s to t we get,

$$\int_{0}^{\infty} \frac{\sin^2 t}{t^2} dt = \pi/2 \quad (\text{or}) \quad \int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \pi/2.$$

Hence the problem.

④ Find the Fourier transform of $~~f(x) = \frac{a-|x|}{}~~$

$$f(x) = \begin{cases} a-|x| & \text{for } |x|<a \\ 0 & \text{for } |x| \geq a \end{cases}$$

Hence deduce that (i) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \pi/2$ and

(ii) $\int_0^\infty \frac{\sin^4 t}{t^4} dt = \frac{\pi}{3}$.

Soln:

Fourier transform of $f(x)$ is

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx \qquad |x|<a \;\Rightarrow\; -a<x<a$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} [a-|x|] e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} [a-|x|][\cos sx + i \sin sx] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} [a-|x|] \cos sx \, dx + \frac{i}{\sqrt{2\pi}} \int_{-a}^{a} [a-|x|] \sin sx \, dx$$

$(a-|x|)\cos sx$ is an even function of x.

$(a-|x|)\sin sx$ is an odd function of x.

Since $|x|$ is an even function, so we can write

$|x| = x$ for $x>0$.

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} [a-|x|] \cos sx \, dx.$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a (a-x) \cos sx \, dx.$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^a (a-x) \cos sx \, dx$$

$\int u \, v \, dx = u v_1 - u' v_2 + u'' v_3 - \dots$

$$= \sqrt{\frac{2}{\pi}}\left[\left[(a-x)\frac{\sin sx}{s}\right]_0^a - \int_0^a\left(\frac{\sin sx}{s}\right)(-1)\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[0 + \left(-\frac{\cos sx}{s^2}\right)_0^a\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{\cos sa}{s^2} + \frac{\cos 0}{s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1-\cos sa}{s^2}\right]$$

$\sqrt{\frac{2}{\pi}}\left((a-x)\left(\frac{\sin sx}{s}\right) - (-1)\right)$

$\sqrt{\frac{2}{\pi}}\left[0 - \frac{\cos sa}{s^2}\right] - \left[0 - \frac{1}{s^2}\right]$

$\sqrt{\frac{2}{\pi}}\left(-\frac{\cos sa}{s^2} + \frac{1}{s^2}\right)$

$= \frac{\sqrt{2}}{\sqrt{\pi}}\left(\frac{1-\cos sa}{s^2}\right)$

Deduction:-

i) By Inversion formula

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos sa}{s^2}\right](\cos sx - i\sin sx)\, ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{1-\cos sa}{s^2}\right][\cos sx - i\sin sx]\, ds.$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1-\cos sa}{s^2}\right)\cos sx\, ds - \frac{i}{\pi}\int_{-\infty}^{\infty}\left(\frac{1-\cos sa}{s^2}\right)\sin sx\, ds.$$

$\left(\frac{1-\cos sa}{s^2}\right)\cos sx$ is an even function

and $\left(\frac{1-\cos sa}{s^2}\right)\sin sx$ is an odd function.

$$\therefore f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1-\cos sa}{s^2}\right)\cos sx - \frac{i}{\pi}(0)$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\left(\frac{1-\cos sa}{s^2}\right)\cos sx\, ds.$$

$1-\cos 2\theta = 2\sin^2\theta$

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{2\sin^2\frac{sa}{2}}{s^2}\cos sx\, ds. \quad \left[\because 1-\cos\theta = 2\sin^2\theta/2\right]$$

$$f(x) = \frac{4}{\pi}\int_0^\infty \frac{\sin^2 \frac{sa}{2}}{s^2}\cos sx\, ds. \longrightarrow ①.$$

i) To find $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt$

put $x=0$ in ① we've

$$f(0) = \frac{4}{\pi}\int_0^\infty \frac{\sin^2 sa/2}{s^2}\cos s(0)\, ds$$

$$a = \frac{4}{\pi}\int_0^\infty \frac{\sin^2\left(\frac{sa}{2}\right)}{s^2}\, ds$$

put $\frac{sa}{2} = t$

$s = \frac{2t}{a} \Rightarrow ds = \frac{2dt}{a}$ $\quad s=0 \Rightarrow t=0$, $s=\infty \Rightarrow t=\infty$

$$\frac{4}{\pi}\int_0^\infty \frac{\sin^2 t}{\left(\frac{2t}{a}\right)^2}\cdot\frac{2}{a}dt = a$$

$$\frac{8}{\pi}\int_0^\infty \frac{\sin^2 t}{4t^2/a^2}\frac{dt}{a} = a$$

$$\frac{2\cancel{a}}{\pi}\int_0^\infty \frac{\sin^2 t}{t^2}dt = \cancel{a}$$

$$\Rightarrow \boxed{\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \pi/2.}$$

(ii) To find $\int_0^\infty \frac{\sin^4 t}{t^4}dt$

By parseval's identity

$$\int_{-\infty}^\infty |F(s)|^2 ds = \int_{-\infty}^\infty |f(x)|^2 dx$$

$$\int_{-\infty}^\infty \left[\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos sa}{s^2}\right]\right]^2 ds = \int_{-a}^a [a-|x|]^2 dx$$

$$\frac{2}{\pi}\int_{-\infty}^\infty \left(\frac{1-\cos sa}{s^2}\right)^2 ds = \int_{-a}^a (a-x)^2 dx$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\left(2\sin^2\frac{sa}{2}\right)^2}{s^4}\right)ds = \int_{-a}^{a}(a^2+x^2-2ax)\,dx$$

$$\frac{8}{\pi}\int_{-\infty}^{\infty}\frac{\sin^4\frac{sa}{2}}{s^4}\,ds = \int_{-a}^{a}(a^2+x^2-2ax)\,dx.$$

$$\frac{8}{\pi}\cdot 2\int_{0}^{\infty}\frac{\sin^4\frac{sa}{2}}{s^4}\,ds = 2\int_{0}^{a}(a^2+x^2-2ax)\,dx$$

since $|x|$ is a even functio[n]

$$\frac{8}{\pi}\int_{0}^{\infty}\frac{\sin^4\frac{sa}{2}}{s^4}\,ds = \int_{0}^{a}(a^2+x^2-2ax)\,dx$$

$$= \left[a^2x+\frac{x^3}{3}-\frac{\cancel{2}ax^2}{\cancel{2}}\right]_0^a$$

$$= \left[a^3+\frac{a^3}{3}-a^3-0\right]$$

$$\frac{8}{\pi}\int_{0}^{\infty}\frac{\sin^4 sa/2}{s^4}\,ds = \frac{a^3}{3}$$

$$\frac{8}{\pi}\int_{0}^{\infty}\frac{\sin^4\frac{sa}{2}}{s^4}\,ds = \frac{a^3}{3}.$$

put $\frac{sa}{2}=t$

$s=\frac{2t}{a}$

$ds=\frac{2}{a}dt$

$$\frac{8}{\pi}\int_{0}^{\infty}\frac{\sin^4 t}{\left(\frac{2t}{a}\right)^4}\,\frac{2}{a}dt = \frac{a^3}{3}$$

$$\frac{8}{\pi}\times\frac{a^4\times 2}{16a}\int_{0}^{\infty}\frac{\sin^4 t}{t^4}\,dt = \frac{a^3}{3}$$

$$\frac{a^3}{\pi}\int_{0}^{\infty}\frac{\sin^4 t}{t^4}\,dt = \frac{a^3}{3}$$

$$\boxed{\int_{0}^{\infty}\frac{\sin^4 t}{t^4}\,dt = \frac{\pi}{3}.}$$

Hence the Problem.

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⑤ Find the Fourier transform of $f(x) = \begin{cases} 1-x^2 & |x|<1 \\ 0 & , |x|>1 \end{cases}$

and hence evaluate i) $\displaystyle\int_0^\infty \frac{\sin t - t\cos t}{t^3}\, dt$

(ii) $\displaystyle\int_0^\infty \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds.$

(iii) $\displaystyle\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right) \cos\frac{x}{2}\, dx$

(iv) $\displaystyle\int_0^\infty \frac{(\sin t - t\cos t)^2}{t^6}\, dt$ $\rightarrow \dfrac{(\sin t - t\cos t)^2}{}$

Soln: Fourier transform of $f(x)$ is

$$F(S) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)(\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\sin sx\, dx$$

(even) (odd)

$$= \frac{1}{\sqrt{2\pi}}\, 2\int_0^1 (1-x^2) \cos sx\, dx + \frac{i}{\sqrt{2\pi}}(0)$$

$$\sqrt{\frac{2}{\pi}} \int_0^1 (1-x^2)\cos sx\, dx.$$

$\int u v\, dx = u v_1 - u' v_2 + u'' v_3 - \cdots$

$= \left[(1-x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(-\frac{\sin sx}{s^3}\right)\right]$

$$= \sqrt{\frac{2}{\pi}} \left[\left[(1-x^2)\frac{\sin sx}{s}\right]_0^1 - \int_0^1 \frac{\sin sx}{s}(-2x)\, dx\right]$$

$= \left[0 - \frac{2\cos s}{s^2} + \frac{2\sin s}{s^3} - 0\right]$

$$= \sqrt{\frac{2}{\pi}} \left[0 + 2\int_0^1 \frac{x\sin sx}{s}\, dx\right]$$

$= \dfrac{2\sin s - 2s\cos s}{s^3}$

$$= \sqrt{\frac{2}{\pi}} \times \int_0^1 \frac{2x\sin sx}{s}\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(2x\left(-\frac{\cos sx}{s^2}\right)\right)_0^1 - \int_0^1 (2)\left(-\frac{\cos sx}{s^2}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{2\cos s}{s^2} + \left[\frac{2\sin sx}{s^3}\right]_0^1\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{2\cos s}{s^2} + \frac{2\sin s}{s^3}\right]$$

$$= 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s}{s^3} - \frac{\cos s}{s^2}\right]$$

$$F(s) = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]$$

Deduction:-

By inversion formula.

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2.\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right](\cos sx - i\sin sx)\, ds$$

$f(s) = \frac{\sin s - s\cos s}{s^3}$

$f(-s) = \frac{-\sin s - (-s)\cos s}{-s^3}$

$= f(s)$

even

$$= \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)(\cos sx - i\sin sx)\, ds$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\, ds \quad \text{(even)}$$

$$- \frac{2i}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\sin sx\, ds \quad \text{(odd)}$$

$$= \frac{2}{\pi}.2\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\, ds - \frac{2i}{\pi}(0)$$

$$f(x) = \frac{4}{\pi}\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\, ds.$$

To find i) $\int_0^\infty \frac{\sin t - t\cos t}{t^3}\,dt$

$$\frac{\pi}{4} f(x) = \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx\, ds.$$

$$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx\, ds = \frac{\pi}{4} f(x).$$

put $x=0$ $\quad f(x) = 1-x^2$

$$\int_0^\infty \frac{\sin s - s\cos s}{s^3}\, ds = \frac{\pi}{4}(1-0)$$

$$\int_0^\infty \frac{\sin s - s\cos s}{s^3}\, ds = \frac{\pi}{4}.$$

i) Changing s to t

$$\boxed{\int_0^\infty \frac{\sin t - t\cos t}{t^3} = \frac{\pi}{4}.}$$

(ii) To find $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right) \cos\frac{x}{2}\, dx$

(ii)

$$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx\, ds = \frac{\pi}{4} f(x).$$

put $x = 1/2$.

$$\int_0^\infty \frac{\sin s - s\cos s}{s^3} \cos\frac{s}{2}\, ds = \frac{\pi}{4}\left[1-\left(\frac{1}{2}\right)^2\right]$$

$$\int_0^\infty \frac{\sin s - s\cos s}{s^3} \cos\frac{s}{2}\, ds = \frac{\pi}{4}\left[1-\frac{1}{4}\right].$$

$$= \frac{\pi}{4}\left[\frac{4-1}{4}\right]$$

$$= \frac{\pi}{4}\left[\frac{3}{4}\right]$$

$$\boxed{\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right) \cos\frac{s}{2}\, ds = \frac{3\pi}{16}}$$

changing s to x

$$\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)\cos\frac{x}{2}\,dx = \frac{3\pi}{16}$$

$$\int_0^\infty \frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}\,dx = -\frac{3\pi}{16}.$$

(iii) To find $\int_0^\infty \frac{(\sin t - t\cos t)^2}{t^6}\,dt$

By parseval's identity, we have

$$\int_{-\infty}^\infty |F(s)|^2 ds = \int_{-\infty}^\infty |f(x)|^2 dx,$$

$$\int_{-\infty}^\infty \left[2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]\right]^2 ds = \int_{-1}^1 (1-x^2)^2 dx.$$

$$\frac{8}{\pi}\int_{-\infty}^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\int_0^1 (1+x^4-2x^2)\,dx$$

$$\frac{8}{\pi}\times 2\int_0^\infty \frac{(\sin s - s\cos s)^2}{s^6}\,ds = 2\int_0^1 (1+x^4-2x^2)\,dx$$

$$\frac{8}{\pi}\int_0^\infty \frac{(\sin s - s\cos s)^2}{s^6}\,ds = \left[x + \frac{x^5}{5} - \frac{2x^3}{3}\right]_0^1$$

$\frac{15+3-10}{15} = \frac{8}{15}$

$$= 1 + \frac{1}{5} - \frac{2}{3}.$$

$$\frac{8}{\pi}\int_0^\infty \frac{(\sin s - s\cos s)^2}{s^6}\,ds = \frac{6}{5} - \frac{2}{3} = \frac{18-10}{15} = \frac{8}{15}$$

$$\int_0^\infty \frac{(\sin s - s\cos s)^2}{s^6}\,ds = \frac{\pi}{15}$$

Changing s to t,

$$\boxed{\int_0^\infty \frac{(\sin t - t\cos t)^2}{t^6}\,dt = \frac{\pi}{15}}$$

⑥ Find the Fourier transform of $e^{-a^2x^2}$, $a>0$. Hence show that $e^{-x^2/2}$ is self reciprocal under fourier transform.

Soln:

Fourier transform of $f(x) = e^{-a^2x^2}$ is

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2}\, e^{isx}\, dx$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[a^2x^2 - \frac{2isax}{2a} + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[ax - \frac{is}{2a}\right]^2 + \left(\frac{is}{2a}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} e^{-\frac{s^2}{4a^2}}\, dx$$

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx.$$

put $u = ax - \frac{is}{2a}$

$du = a\,dx$

$dx = \frac{du}{a}$

x	$-\infty$	∞
u	$-\infty$	∞

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{a}$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} du.$$

$$= \frac{2\, e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_{0}^{\infty} e^{-u^2} du$$

put $t = u^2$ | when $u=0 \Rightarrow t=0$

$\Rightarrow u = \sqrt{t}$ | $u=\infty \Rightarrow t=\infty$.

$dt = 2u\,du$

$du = \frac{dt}{2u} = \frac{dt}{2\sqrt{t}}$

$$= \frac{\cancel{2}\, e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_0^\infty e^{-t} \frac{dt}{\cancel{2}\sqrt{t}}$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_0^\infty e^{-t}\, t^{-1/2}\, dt$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_0^\infty e^{-t}\, t^{1/2-1}\, dt$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \Gamma\left(\tfrac{1}{2}\right)$$

$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$

$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$

$$= \frac{e^{-s^2/4a^2} \times \cancel{\sqrt{\pi}}}{a\sqrt{2}\,\cancel{\sqrt{\pi}}}$$

$$F(s) = \frac{e^{-s^2/4a^2}}{a\sqrt{2}}$$

$$F\left(e^{-a^2x^2}\right) = \frac{1}{a\sqrt{2}}\, e^{-s^2/4a^2} \longrightarrow (1)$$

put $a = \frac{1}{\sqrt{2}}$

$$F\left[e^{-x^2/2}\right] = \frac{1}{\frac{1}{\sqrt{2}}\cdot\sqrt{2}}\left[e^{-\frac{s^2}{4(1/2)}}\right]$$

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$F[e^{-x^2/2}] = [e^{-s^2/2}]$

$F[e^{-x^2/2}] = e^{-s^2/2}$, the function

$f(x) = e^{-x^2/2}$ is self reciprocal under the Fourier transform.

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S.T the F.T of $f(x) = \begin{cases} a^2-x^2, & |x|<a \\ 0 & |x| \geq a \end{cases}$

is $2\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa - sa\cos sa}{s^3}\right]$

Hence deduce that $\int_0^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)dt = \pi/4$.

Soln: The fourier transform of $f(x)$ is

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{\infty} f(x)\, e^{isx}\, dx$$

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2-x^2)(\cos sx + i\sin sx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2-x^2)\cos sx\,dx + \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(a^2-x^2)\sin sx\,dx$$

(even) (odd)

$$= \frac{1}{\sqrt{2\pi}}\cdot 2\int_0^a (a^2-x^2)\cos sx\,dx + 0$$

$$= \sqrt{\frac{2}{\pi}}\int_0^a (a^2-x^2)\cos sx\,dx.$$

$$= \sqrt{\frac{2}{\pi}}\left[(a^2-x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - \frac{2a\cos sa}{s^2} + \frac{2\sin sa}{s^3} - 0 - 0 - 0\right]$$

$$= \sqrt{\frac{2}{\pi}}\cdot 2\left[-\frac{a\cos sa}{s^2} + \frac{\sin sa}{s^3}\right]$$

$$\boxed{F(s) = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa - sa\cos sa}{s^3}\right]}$$

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Deduction:

By Inversion formula

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[2\sqrt{\frac{2}{\pi}}\left(\frac{\sin sa - sa\cos sa}{s^3}\right)\right]\left[\cos sx - i\sin sx\right] ds$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin sa - sa\cos sa}{s^3}\right)\cos sx\, ds$$

(even)

$$- \frac{2i}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin sa - sa\cos sa}{s^3}\right)\sin sx\, ds$$

(odd).

$$= \frac{2}{\pi}\cdot 2\int_{0}^{\infty}\left(\frac{\sin sa - sa\cos sa}{s^3}\right)\cos sx\, ds - 0.$$

$$f(x) = \frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin sa - sa\cos sa}{s^3}\right)\cos sx\, ds$$

$\frac{\sin sa - sa\cos sa}{s^3}$ is even.

$$\int_{0}^{\infty}\frac{\sin sa - sa\cos sa}{s^3}\cos sx\, ds = \frac{\pi}{4} f(x) \longrightarrow (1)$$

put $x=0$ in (1) we get

$$\int_{0}^{\infty}\left(\frac{\sin sa - sa\cos sa}{s^3}\right)\cos s(0)\, ds = \frac{\pi}{4}(a^2-0).$$

$$\int_{0}^{\infty}\left(\frac{\sin sa - sa\cos sa}{s^3}\right) ds = a^2\left(\frac{\pi}{4}\right)$$

put $sa = t$

$\Rightarrow s = t/a$

$ds = dt/a$.

$$\therefore \int_0^\infty \frac{\sin t - t\cos t}{(t/a)^3} \cdot \frac{dt}{a} = a^2\left(\frac{\pi}{4}\right).$$

$$a^2 \int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right) dt = a^2\left(\frac{\pi}{4}\right)$$

$$\boxed{\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right) dt = \frac{\pi}{4}}$$

Properties of Fourier Transform:-

Change of scale property:-

If $F[f(x)] = F(s)$ then $F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right), a>0$

Soln: W.K.T $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx.$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{isx} dx$$

put $t = ax$ | when $x = -\infty$, $t = -\infty$

$dt = a\,dx$ | when $x = \infty$, $t = \infty$.

$\Rightarrow x = t/a$, $dx = \frac{dt}{a}$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{is\left(\frac{t}{a}\right)} \frac{dt}{a}$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\left(\frac{s}{a}\right)t} dt$$

$$\boxed{F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right)}, \quad a>0$$

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Shifting theorem:-

If $F[f(x)] = F(s)$ then

i) $F[f(x-a)] = e^{ias} F(s)$

(ii) $F[e^{iax} f(x)] = F(s+a)$

Soln: W.K.T $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx = F(s)$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} dx.$$

put $t = x-a$ | $\Rightarrow x = a+t$

$dt = dx$ | when $x = -\infty$, $t = -\infty$

when $x = \infty$, $t = \infty$.

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{is(t+a)} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} e^{isa} dt$$

$$= e^{ias} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt$$

i) $F[f(x-a)] = e^{ias} F(s)$.

(ii) $F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{isx} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx.$$

$$F[e^{iax} f(x)] = F(s+a)$$

Modulation property:-

If $F[f(x)] = F(s)$, then

$$F[f(x)\cos ax] = \frac{1}{2}[F(s+a) + F(s-a)]$$

Soln: $$F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos ax\, e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left(\frac{e^{iax}+e^{-iax}}{2}\right)e^{isx}\,dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{iax}e^{isx}\,dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iax}e^{isx}\,dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s+a)x}\,dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{i(s-a)x}\,dx\right]$$

$$\boxed{F[f(x)\cos ax] = \frac{1}{2}[F(s+a) + F(s-a)]}$$

Note: $$F[f(x)\sin ax] = \frac{1}{2i}[F(s+a) - F(s-a)]$$

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Fourier sine and Cosine Transform:-

Fourier sine Transform (F.S.T)

The infinite fourier sine transform of $f(x)$ is defined by

$$F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx.$$

Inverse fourier sine Transform:-

The inverse fourier sine transform of $F_s(s)$ is defined by

$$F_s^{-1}[F_s(s)] = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds.$$

Fourier Cosine Transform:- [FCT]

The infinite fourier Cosine transform of $f(x)$ is defined by

$$F_c[f(x)] = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx.$$

Inverse fourier Cosine Transform:-

The inverse fourier Cosine Transform of $F_c(s)$ is defined by

$$F_c^{-1}[F_c(s)] = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds.$$

Properties of FST and FCT

1) Linear Property:-

$$F_s[af(x)+bg(x)] = aF_s[f(x)] + bF_s[g(x)]$$

$$F_c[af(x)+bg(x)] = aF_c[f(x)] + bF_c[g(x)]$$

2) Modulation property:-

1. $F_c[f(x)\cos ax] = \frac{1}{2}\left[F_c(s+a) + F_c(s-a)\right]$

2. $F_c[f(x)\sin ax] = \frac{1}{2}\left[F_s(a+s) + F_s(a-s)\right]$

3. $F_s[f(x)\cos ax] = \frac{1}{2}\left[F_c[s+a] + F_c[s-a]\right]$

4. $F_s[f(x)\sin ax] = \frac{1}{2}\left[F_c(s-a) - F_c(s+a)\right]$.

3) Change of scale property:-

i) $F_c[f(ax)] = \frac{1}{a} F_c(s/a)$

(ii) $F_s[f(ax)] = \frac{1}{a} F_s(s/a)$

4) F.T of Derivatives:-

i) $F_c[f'(x)] = sF_s(s) - \sqrt{\frac{2}{\pi}}\, f(0)$ &

Privided: $f(x) \to 0$ as $x \to \infty$

(ii) $F_s[f'(x)] = -sF_s(s)$ provided $f(x) \to 0$ as $x \to \infty$

5) $F_s[x f(x)] = -\frac{d}{ds} F_c[f(x)]$ and

$$F_c[x f(x)] = \frac{d}{ds} F_s[f(x)].$$

Parseval's identities and other Related identities:-

FCT

i) If $F_c(s)$ and $G_c(s)$ are fourier cosine Transforms of $f(x)$ and $g(x)$ respectively. then $\int_0^\infty F_c(s)\, G_c(s)\, ds = \int_0^\infty f(x) g(x)\, dx$.

(ii) If $F_c(s)$ is F.C.T of $f(x)$, then

$$\int_0^\infty [F_c(s)]^2 ds = \int_0^\infty [f(x)]^2 dx.$$

FST

(iii) If $F_s(s)$ and $G_s(s)$ are fourier sine transforms of $f(x)$ and $g(x)$ resply then

$$\int_0^\infty F_s(s)\, G_s(s)\, ds = \int_0^\infty f(x) g(x)\, dx.$$

(iv) If $F_s(s)$ is F.S.T of $f(x)$, then

$$\int_0^\infty [F_s(s)]^2 ds = \int_0^\infty [f(x)]^2 dx.$$

① Find the Fourier Cosine Transform of $e^{-ax}, a>0$

Soln:

$$F_c[f(x)] = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{(-a)^2 + s^2} \left[-a\cos sx + s\sin sx \right] \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-\infty}}{a^2+s^2} - \left(\frac{e^0}{s^2+a^2} \left(-a\cos s(0) + s \sin s(0) \right) \right) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{1}{s^2+a^2} (-a) \right]$$

$$\boxed{F_c(s) = \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2} \right]}$$

$e^{-\infty} = 0$

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2} [a\cos bx + b\sin bx]$$

② Find the Fourier sine transform of $e^{-ax}, a>0$.

Soln:

$$F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx\, dx$$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{s^2+a^2} \left[-a\sin sx - s\cos sx \right] \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{e^0}{s^2+a^2} \left(-a\sin s(0) - s\cos s(0) \right) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-1}{s^2+a^2} (-s) \right]$$

$$\boxed{F_s(s) = \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right]}$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} \left[a \sin bx - b \cos bx\right]$$

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(5) Find the FST of e^{-x}. Hence S.T

$$\int_0^\infty \frac{x \sin mx}{1+x^2}\,dx = \frac{\pi}{2} e^{-m}, \quad m>0.$$

Soln: W.K.T $F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$.

$$F_s(e^{-x}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx \, dx.$$

$$\boxed{F_s(s) = F_s(e^{-x}) = \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+1}\right]}$$

By Inversion formula,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$$

$$e^{-x} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+1}\right) \sin sx \, ds$$

$$e^{-x} = \frac{2}{\pi} \int_0^\infty \left(\frac{s}{s^2+1}\right) \sin sx \, ds$$

$$\int_0^\infty \left(\frac{s}{s^2+1}\right) \sin sx \, ds = \frac{\pi}{2} e^{-x}$$

Changing s to x and x to m we get

$$\boxed{\int_0^\infty \frac{x}{1+x^2} \sin mx \, dx = \frac{\pi}{2} e^{-m}}$$

(10) Find the FCT of $\frac{e^{-ax}}{x}$ and hence find FCT of $\frac{e^{-ax}-e^{-bx}}{x}$.

Soln:

$$F_c[s] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos sx\,dx.$$

$$F_c[s] = F_c\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos sx\,dx$$

D. w.r.to 's' on bothsides,

$$\frac{d}{ds}\left[F_c(s)\right] = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos sx\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{d}{ds}\left(\frac{e^{-ax}}{x}\cos sx\right)dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}(-\sin sx\cdot x)\,dx$$

$$\frac{d}{ds}\left[F_c(s)\right] = -\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$\frac{d}{ds}\left[F_c(s)\right] = -\sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+a^2}\right]$$

Integrating w.r.to 's' on bothsides,

$$F_c(s) = -\sqrt{\frac{2}{\pi}}\int \frac{s\,ds}{s^2+a^2}$$

$$= -\sqrt{\frac{2}{\pi}}\cdot\frac{1}{2}\int\frac{2s\,ds}{s^2+a^2}$$

$$= -\sqrt{\frac{2}{\pi}}\cdot\frac{1}{2}\log(s^2+a^2).$$

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$$\boxed{F_c\left[\frac{e^{-ax}}{x}\right] = F_c(s) = -\frac{1}{\sqrt{2\pi}}\log(s^2+a^2)}$$

lly

$$F_c\left[\frac{e^{-bx}}{x}\right] = \frac{-1}{\sqrt{2\pi}}\log(s^2+b^2).$$

$$\therefore F_c\left[\frac{e^{-ax}-e^{-bx}}{x}\right] = F_c\left[\frac{e^{-ax}}{x}\right] - F_c\left[\frac{e^{-bx}}{x}\right]$$

$$= \frac{-1}{\sqrt{2\pi}}\log(s^2+a^2) - \left(-\frac{1}{\sqrt{2\pi}}\log(s^2+b^2)\right)$$

$$= \frac{-1}{\sqrt{2\pi}}\log(s^2+a^2) + \frac{1}{\sqrt{2\pi}}\log(s^2+b^2)$$

$$= \frac{1}{\sqrt{2\pi}}\left[\log(s^2+b^2) - \log(s^2+a^2)\right]$$

$$F_c\left[\frac{e^{-ax}-e^{-bx}}{x}\right] = \frac{1}{\sqrt{2\pi}}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

Define Convolution of two functions:-

If $f(x)$ and $g(x)$ are two functions defined in $(-\infty, \infty)$. then the convolution of $f(x)$ and $g(x)$ is defined by

$$(f*g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

Convolution Theorem:-

The fourier transform of the Convolution of $f(x)$ and $g(x)$ is the product of their fourier transforms.

ie) $F[f(x) * g(x)] = F[f(x)]\, F[g(x)]$

$= F(s)G(s).$

(or)

If $F[f(x)] = F(s)$ and $F[g(x)] = G(s)$ then

$$\boxed{F[f(x) * g(x)] = F[f(x)]\, F[g(x)] = F(s)G(s).}$$

① $F_s[x f(x)] = -\frac{d}{ds}[F_c(s)]$

Soln:-

WKT $F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$

Diff both sides w.r.to s

$$\frac{d}{ds}[F_c(f(x))] = \sqrt{\frac{2}{\pi}} \frac{d}{ds} \int_0^\infty f(x) \cos sx \, dx$$

$$\frac{d}{ds}[F_c(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \frac{\partial}{\partial s}(\cos sx)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) (-\sin sx) \cdot x \, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\, x \sin sx \, dx$$

$$= -F_s[x f(x)]$$

(ie) $F_s[x f(x)] = -\frac{d}{ds}[F_c(s)].$

② $F_c[x f(x)] = \frac{d}{ds} F_s(s)$

Soln

WKT

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

Diff both sides w.r.to 's' we get

$$\frac{d}{ds} F_s(s) = \sqrt{\frac{2}{\pi}} \frac{d}{ds} \int_0^\infty f(x) \sin sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \frac{d}{ds}(\sin sx)\, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) . \cos sx . x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x f(x) \cos sx \, dx$$

$$= F_c[x f(x)].$$

(ii) $\boxed{F_c[x f(x)] = \frac{d}{ds} F_s(s)}$

S.T the F.T of $e^{-x^2/2}$ is $e^{-s^2/2}$ (or)

S.T $e^{-x^2/2}$ is self reciprocal with respect to fourier Transform.

Sol: Given: $f(x) = e^{-x^2/2}$

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2} - isx\right)} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2isx)} dx$$

$a = x$, $2ab = 2isx$ $\qquad (a-b)^2 = a^2 + b^2 - 2ab..$

$\Rightarrow \cancel{2x}b = \cancel{2}i\cancel{x}s$ $\qquad \Rightarrow a^2 - 2ab = (a-b)^2 - b^2$

$b = is$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(x-is)^2 - (is)^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-is)^2} . e^{\frac{(is)^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2} . e^{-s^2/2} dx.$$

$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2isx + (is)^2 - (is)^2)}$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[(x-is)^2 - (is)^2]}$

$$= \frac{e^{-s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2} dx.$$

$u = \frac{x - is}{\sqrt{2}}$

$du = \frac{dx}{\sqrt{2}}$

$dx = \sqrt{2}\, du$

$$= \frac{e^{-s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} . \sqrt{2}\, du$$

$$= \frac{e^{-s^2/2}}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-u^2} du}_{(\sqrt{\pi})} = e^{-s^2/2}$$

Convolution Theorem:-

The Fourier Transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier Transforms.

(ie) $F[f(x)*g(x)] = F(s).G(s) = F[f(x)]\, F[g(x)]$

Proof: WKT $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$

$$F[f(x)*g(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(x)*g(x)]\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt\right] e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t) g(x-t)\, e^{isx}\, dt\, dx$$

By changing the order of integration, we get

$$= \left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, g(x-t)\, e^{isx}\, dx\, dt$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left[\int_{-\infty}^{\infty} g(x-t)\, e^{isx}\, dx\right] dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t)\, e^{isx}\, dx\right] dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, F[g(x-t)]\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, G(s)\, dt$$

By shifting property
$F[f(x-a)] = e^{ia}\ldots$
$= e^{ias} F(s)$

$$= G(s)\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt\right]$$

$$= G(s)\, F(s)$$

$F[f(x)*g(x)] = F(s)\, G(s).$

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Parseval's Identity:-

If $F(s)$ is the Fourier Transform of $f(x)$, then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds.$$

Proof: By convolution Thm,

$$F[f(x) * g(x)] = F(s).G(s)$$

$$f(x) * g(x) = F^{-1}[F(s).G(s)]$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) G(s)\, e^{-isx} ds. \rightarrow (1)$$

put $x=0$

$$\int_{-\infty}^{\infty} f(t)\, g(-t)\, dt = \int_{-\infty}^{\infty} F(s) G(s)\, (1)\, ds \rightarrow (2)$$

$g(-t) = \overline{f(t)}$ then it follows that $G(s) = \overline{F(s)}$

$$(2) \Rightarrow \int_{-\infty}^{\infty} f(t)\, \overline{f(t)}\, dt = \int_{-\infty}^{\infty} F(s) \overline{F(s)}\, ds$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(s)|^2 ds}$$

$|z|^2 = z\bar{z}$

UNIT-III

PARTIAL DIFFERENTIAL EQUATIONS

A partial differential eqns is one which involves partial derivatives. The order a partial differential eqn is the order of the highest derivative occuring in it.

$$z = f(x,y)$$

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = s \;\&\; \frac{\partial^2 z}{\partial y^2} = t$$

$p + qx = x + y$ is a p.d.e of order 1.

and $r + t = x^2 + y$ is a p.d.e of order 2.

Formation of P.D.Eqns By Elimination of Arbitrary Constants :-

Consider an eqn $f(x,y,z,a,b) = 0 \rightarrow ①$

which contains two arbitrary constants a and b

Diff ①. P.w.r.to x & y we get

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0 \rightarrow ②$$

$$\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = 0 \rightarrow ③$$

Eliminating two constants a and b from three eqns we shall obtain an eqn of the form $\phi(x,y,p,q) = 0$, which is the p.d.e of the first order.

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order:

The order of a p.d.e is the order of the highest partial differential coefficient occuring in it

degree:-

The degree of this highest derivative is the degree of the p.d.e.

① Form the p.d.e by eliminating the arbitrary constants from $z = ax + by + a^2 + b^2$

Soln: Given $z = ax + by + a^2 + b^2 \rightarrow ①$

Diff ① p.w.r.to x & y resply.

$$\frac{\partial z}{\partial x} = p = a \rightarrow ②$$

$$\frac{\partial z}{\partial y} = q = b \rightarrow ③$$

sub ② & ③ values in ① we get

$$z = px + qy + p^2 + q^2$$

which is the required p.d.e.

Case (i) If A.C $\leq$ I.V then we get p.d.e of order 1

we use p and q only.

A.C → Arbitrary constants (a, b).

I.V → Independent variable (x, y)

② form the p.d.e by eliminating the arbitrary constants from $z = ax + by + ab$.

Soln. Given $z = ax + by + ab \rightarrow ①$

$p = a$, $q = b$ in ① we get

$$z = px + qy + pq$$

which is the required p.d.e.

3) Form a p.d.e by eliminating the arbitrary constants a and b from $z = (x+a)^2 + (y-b)^2$

Soln: Given $z = (x+a)^2 + (y-b)^2 \longrightarrow (1)$

$$\frac{\partial z}{\partial x} = p = 2(x+a)$$

$$\Rightarrow p = 2(x+a)$$

$$\boxed{x+a = p/2} \longrightarrow (2)$$

$$q = 2(y-b)$$

$$\boxed{\frac{q}{2} = y-b} \longrightarrow (3)$$

(1) $\Rightarrow$ sub (2) & (3) in (1) we get

$$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$$z = \frac{p^2}{4} + \frac{q^2}{4}$$

$$\boxed{4z = p^2 + q^2}$$

4) $z = (x^2+a)(y^2+b)$

$$p = 2x(y^2+b) \Rightarrow \frac{p}{2x} = y^2+b$$

$$q = 2y(x^2+a) \Rightarrow \frac{q}{2y} = x^2+a$$

$$\therefore z = \left(\frac{p}{2x}\right)\left(\frac{q}{2y}\right)$$

$$\boxed{4xyz = pq}$$

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5) $z = ax^3 + by^3$

Soln: Given $z = ax^3 + by^3 \rightarrow (1)$

$$\frac{\partial z}{\partial x} = p = 3ax^2$$

$$\Rightarrow \frac{p}{3x^2} = a \rightarrow (2)$$

$$\frac{\partial z}{\partial y} = q = 3by^2$$

$$\Rightarrow \frac{q}{3y^2} = b \rightarrow (3)$$

sub (2) & (3) in (1) we get

$$z = \left(\frac{p}{3x^2}\right)x^3 + \left(\frac{q}{3y^2}\right)y^3.$$

$$z = \frac{p}{3}x + \frac{q}{3}y.$$

$$3z = px + qy.$$

$$\boxed{px + qy = 3z}$$

6) $z = (x^2+a^2)(y^2+b^2)$

$$p = 2x(y^2+b^2)$$

$$\Rightarrow \frac{p}{2x} = y^2+b^2$$

$$q = 2y(x^2+a^2) \Rightarrow \frac{q}{2y} = x^2+a^2$$

$$\therefore z = \left(\frac{p}{2x}\right)\left(\frac{q}{2y}\right)$$

$$\boxed{4xyz = pq.}$$

⑦ $z = ax^n + by^n$

Soln: Given $z = ax^n + by^n$.

$p = anx^{n-1} \Rightarrow \frac{px}{n} = ax^n$.

$q = bny^{n-1} \Rightarrow \frac{qy}{n} = by^n$

$\therefore z = \left(\frac{px}{n}\right) + \left(\frac{qy}{n}\right)$

$\boxed{nz = px + qy}$

⑧ $z = a^2x + ay^2 + b$

Soln: Given $z = a^2x + ay^2 + b$

$p = a^2$

$q = 2ay$

$\Rightarrow y = \frac{q}{2a}$

$y^2 = \frac{q^2}{4a^2}$

$y^2 = \frac{q^2}{4p} \quad (\because p = a^2)$

$\boxed{4y^2p = q^2}$

⑨ $(x-a)^2 + (y-b)^2 + z^2 = c^2$

Soln: Given $(x-a)^2 + (y-b)^2 + z^2 = c^2 \rightarrow ①$

$2(x-a) + 2zp = 0.$

$x - a = -zp.$

III^{ly} $y - b = -zq$

① $\Rightarrow (-zp)^2 + (-zq)^2 + z^2 = c^2$

$p^2z^2 + q^2z^2 + z^2 = c^2$

$$\boxed{z^2(1+p^2+q^2) = c^2}$$

(10) $z = a(x+y)+b$

Soln: Given $z = a(x+y)+b \rightarrow (1)$

$p = a \rightarrow (2)$

$q = a \rightarrow (3)$

from (2) and (3) we get $p=q$.

(11) $(x-a)^2+(y-b)^2 = z^2\cot^2\alpha$

Soln: Given $(x-a)^2+(y-b)^2 = z^2\cot^2\alpha \rightarrow (1)$

Diff (1) p.w.r. to x we get

$2(x-a) = 2z\frac{\partial z}{\partial x}\cot^2\alpha$

$x-a = zp\cot^2\alpha \rightarrow (2)$

D. (1) p.w.r. to 'y' we get

$2(y-b) = 2z\frac{\partial z}{\partial y}\cot^2\alpha$

$y-b = zq\cot^2\alpha \rightarrow (3)$

Eqns (2) & (3) in (1) we get

$(zp\cot^2\alpha)^2 + (zq\cot^2\alpha)^2 = z^2\cot^2\alpha$

$z^2p^2\cot^4\alpha + z^2q^2\cot^4\alpha = z^2\cot^2\alpha$

$\div z^2\cot^4\alpha$

$p^2+q^2 = \frac{1}{\cot^2\alpha}$

$$\boxed{p^2+q^2 = \tan^2\alpha}$$

(11) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Soln: Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

D.p.w.r.to x

$$\frac{2x}{a^2}+\frac{2zp}{c^2}=0 \rightarrow (1)$$

D.p.w.r.to y

$$\frac{2y}{b^2}+\frac{2zq}{c^2}=0 \rightarrow (2)$$

D.p.w.r.to y

$$0+\frac{2z}{c^2}\frac{\partial^2 z}{\partial y\partial x}+\frac{2p}{c^2}\frac{\partial z}{\partial y}=0$$

$$\boxed{zs+pq=0}$$

(12) $(x-a)^2+(y-b)^2+z^2=1$

Soln: $x-a=-zp$

$y-b=-zq$

$$(-zp)^2+(-zq)^2+z^2=1$$

$$z^2p^2+z^2q^2+z^2=1$$

$$z^2(p^2+q^2+1)=1.$$

(13) Find the p.d.e of all planes having equal intercepts in the x and y axes

Soln: Eqn of the plane having equal x & y intercepts is $\frac{x}{a}+\frac{y}{a}+\frac{z}{b}=1 \rightarrow (1)$

D. (1) p.w.r.to x & y

$$\frac{1}{a}+\frac{p}{b}=0 \;\&\; \frac{1}{a}+\frac{q}{b}=0.$$

$\frac{1}{a}=-\frac{p}{b} \rightarrow (2)$

$\frac{1}{a}=-\frac{q}{b} \rightarrow (3)$

From (2) & (3) $\Rightarrow$ $\frac{p}{c}=\frac{q}{c}$

$$\boxed{p=q}$$

Intercept form

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

where a,b,c are intercepts in x,y,z.

Formation of P.D. Eqns by Elimination of Arbitrary functions:-

The elimination of one arbitrary function from a given relation gives a p.d.e of first order while elimination of two arbitrary function from a given relation gives a second or higher order p.d. eqns.

Case (i) a.f = 1, p.d.e order = 1, we use p & q only

1) Eliminate f from $z = f(x^2 - y^2)$

Soln: Given $z = f(x^2 - y^2) \longrightarrow (1)$

diff (1) p.w.r.t. x & y we get

$$p = \frac{\partial z}{\partial x} = f'(x^2 - y^2)\, 2x \longrightarrow (2)$$

$$q = \frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y) \longrightarrow (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{f'(x^2 - y^2)\, 2x}{f'(x^2 - y^2)(-2y)} = \frac{-x}{y}$$

$$py = -qx$$

$\boxed{py + qx = 0}$ is the required p.d.e

2) Eliminate f from $z = x + y + f(xy)$

Soln: Given $z = x + y + f(xy) \longrightarrow (1)$

$$p = 1 + f'(xy).y \Rightarrow p - 1 = y f'(xy) \rightarrow (2)$$

$$q = 1 + f'(xy).x \Rightarrow q - 1 = x f'(xy) \rightarrow (3)$$

$\frac{②}{③} \Rightarrow \frac{p-1}{q-1} = \frac{yf'(xy)}{xf'(xy)}$

$x(p-1) = y(q-1)$

$xp - x = yq - y$

$\boxed{xp - qy = x - y}$ is the required p.d.e.

③ Eliminate f from $z = f\left(\frac{x}{y}\right)$

Soln:

$p = f'\left(\frac{x}{y}\right) \cdot \left(\frac{1}{y}\right)$

$q = f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right)$

$$\frac{p}{q} = \frac{f'(x/y)\left(\frac{1}{y}\right)}{f'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)} = \frac{\frac{1}{y}}{-x/y^2} = \frac{-y}{x}.$$

$px = -yq \Rightarrow \boxed{px + qy = 0}$

④ Eliminate f from $z = f\left(\frac{xy}{z}\right)$.

Soln: Given $z = f\left(\frac{xy}{z}\right) \longrightarrow ①$

$p = f'\left(\frac{xy}{z}\right) \cdot \frac{\partial}{\partial x}\left(\frac{xy}{z}\right)$

$= f'\left(\frac{xy}{z}\right) y \left[\frac{z(1) - x\frac{\partial z}{\partial x}}{z^2}\right]$

$p = yf'\left(\frac{xy}{z}\right)\left(\frac{z - xp}{z^2}\right) \longrightarrow ②$

$q = f'\left(\frac{xy}{z}\right) x \frac{\partial}{\partial y}\left(\frac{y}{z}\right)$

$q = xf'\left(\frac{xy}{z}\right)\left[\frac{z - y\frac{\partial z}{\partial y}}{z^2}\right]$

$q = xf'\left(\frac{xy}{z}\right)\left[\frac{z - yq}{z^2}\right] \longrightarrow ③$

$\frac{②}{③} \Rightarrow \frac{p}{q} = \frac{y}{x}\left(\frac{z - xp}{z - yq}\right)$

$$px(z-yq) = qy(z-xp)$$

$$pxz - pq\cancel{xy} = qyz - \cancel{xypq}$$

$\boxed{px = qy}$ is the required p.d.e.

4. from the PDE by eliminating the arbitrary function from $\phi\left[z^2-xy, \frac{x}{z}\right]=0$.

Soln: Given $\phi\left[z^2-xy, \frac{x}{z}\right]=0$.

$$\text{ie)}\quad z^2 - xy = \phi\left(\frac{x}{z}\right) \longrightarrow (1)$$

D.(1) p.w.r.to 'x'

$$2z\frac{\partial z}{\partial x} - y = \phi'\left(\frac{x}{z}\right)\left[\frac{z(1) - x\frac{\partial z}{\partial x}}{z^2}\right]$$

$$2zp - y = \phi'\left(\frac{x}{z}\right)\left[\frac{z-xp}{z^2}\right] \longrightarrow (2)$$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0.$$

$$\Rightarrow \begin{vmatrix} 2zp-y & \frac{z-px}{z^2} \\ 2zq-x & -\frac{xq}{z^2} \end{vmatrix} = 0$$

$$\Rightarrow px^2 - q(xy - 2z^2) = xz$$

Form the pde by eliminating the arbitrary functions from

1. $z = x^2 f(y) + y^2 g(x)$
2. $z = f(x+2y) + x g(x+2y)$
3. $z = f(2x-3y) + x g(2x-3y)$

Soln:

(1) $z = x^2 f(y) + y^2 g(x) \rightarrow (1)$

$p = \frac{\partial z}{\partial x} = 2x f(y) + y^2 g'(x) \rightarrow (2)$

$q = \frac{\partial z}{\partial y} = x^2 f'(y) + 2y g(x) \rightarrow (3)$

$r = \frac{\partial^2 z}{\partial x^2} = 2f(y) + y^2 g''(x) \rightarrow (4)$

$t = \frac{\partial^2 z}{\partial y^2} = x^2 f''(y) + 2g(x) \rightarrow (5)$

$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[x^2 f'(y) + 2y g(x)\right]$

$s = 2x f'(y) + 2y g'(x) \rightarrow (6)$

$(2) \times x \Rightarrow px = 2x^2 f(y) + xy^2 g'(x)$

$(3) \times y \Rightarrow qy = x^2 y f'(y) + 2y^2 g(x)$

$px + qy = 2x^2 f(y) + xy^2 g'(x) + x^2 y f'(y) + 2y^2 g(x)$

$= 2\left[x^2 f(y) + y^2 g(x)\right] + xy\left[x f'(y) + y g'(x)\right]$

$px + qy = 2[z] + xy\left[\frac{s}{2}\right]$

$px + qy = \frac{4z + xys}{2}$

$2px + 2qy = 4z + xys$

$\boxed{4z = 2px + 2qy - sxy}$ which is required p.d.e.

(4)

$$z = f(x+t) + g(x-t) \rightarrow (1)$$

$$p = \frac{\partial z}{\partial x} = f'(x+t) + g'(x-t) \rightarrow (2)$$

$$q = \frac{\partial z}{\partial t} = f'(x+t) - g'(x-t) \rightarrow (3)$$

$$\frac{\partial^2 z}{\partial x^2} = r \Rightarrow f''(x+t) + g''(x-t) \rightarrow (4)$$

$$\frac{\partial^2 z}{\partial t^2} = t \Rightarrow f''(x+t) + g''(x-t) \rightarrow (5)$$

From (4) & (5) we get

$$\boxed{\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2}}$$

Solution of P.d.e in ordinary cases:-

(1) Find the general soln of $\frac{\partial^2 z}{\partial y^2} = 0$

Soln: Given $\frac{\partial^2 z}{\partial y^2} = 0$.

(ie) $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = 0$.

Integrating w.r.t y on both sides,

$$\frac{\partial z}{\partial y} = a \text{ (constant)}.$$

ie) $\frac{\partial z}{\partial y} = f(x)$

Again Integrating w.r.to y on both sides

$$z = f(x)y + b$$

$$z = f(x)y + F(x)$$

$z = yf(x) + F(x)$ where both $f(x)$ & $F(x)$ are arbitrary.

② Solve $\frac{\partial^2 z}{\partial x \partial y} = 0$

Soln: Given $\frac{\partial^2 z}{\partial x \partial y} = 0$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 0$$

Integrating w.r.t x, we get

$$\frac{\partial z}{\partial y} = f(y) \text{ (constant)}$$

Again Integrating w.r.t y we get

$$z = \int f(y)\,dy + \phi(x)$$

$$z = F(y) + \phi(x)$$

③ Solve $\frac{\partial^2 z}{\partial x^2} = \sin y$

Soln:
$$\frac{\partial^2 z}{\partial x^2} = \sin y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \sin y$$

Integrating w.r.t x we get

$$\frac{\partial z}{\partial x} = (\sin y)x + f(y)$$

Again Integrating w.r.t x we get

$$z = (\sin y)\frac{x^2}{2} + f(y)x + F(y)$$

$$z = \frac{x^2}{2}\sin y + xf(y) + F(y)$$

where both $f(y)$ & $F(y)$ are arbitrary.

(4) Solve $\frac{\partial z}{\partial x} = \sin x$

Soln:

$$\frac{\partial z}{\partial x} = \sin x$$

I. p. w. r. to x

$$z = -\cos x + f(y)$$

where $f(y)$ is an arbitrary function.

Lagrange's Linear Equation:-

$Pp + Qq = R$

The eqn of the form $Pp + Qq = R$ where P, Q, R are functions of x, y, z is called Lagrange's linear eqn.

Here the partial derivatives $p = z_x$ & $q = z_y$ occur only in first degree and are not multiplied together.

To solve the eqn $Pp + Qq = R$

Working Rule:-

Step: 1 form the auxiliary eqns $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ → (1)

Step: 2 Solve the auxiliary eqns by the following two methods

i) Method of grouping

(ii) Method of multipliers.

Method of grouping:-

* First take any two functions from the A.E's and solve them we get the soln $u(x,y) = c_1$

* Again take another two directions & solve them we get the soln $v(x,y) = c_2$.

* Now we've the two solns $u = c_1$ & $v = c_2$.

* These constitute the required soln $\phi(u,v) = 0$.

Pbm:

① Find the soln of $px^2 + qy^2 = z^2$

Soln: This eqn is of the form $Pp + Qq = R$
Here $P = x^2$, $Q = y^2$, $R = z^2$.

The A.E is $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$

$\frac{x^{-2+1}}{-2+1} = -\frac{1}{x}$

Step:1

$$\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$$

$$\frac{-1}{x} = \frac{-1}{y} + c_1$$

$$\frac{1}{y} - \frac{1}{x} = c_1$$

Step:2

$$\int \frac{dy}{y^2} = \int \frac{dz}{z^2}$$

$$\frac{-1}{y} = \frac{-1}{z} + c_2$$

$$\frac{1}{z} - \frac{1}{y} = c_2$$

∴ The general soln is $\phi\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{z} - \frac{1}{y}\right) = 0$.

② Solve $px + qy = z$.

Soln: Given $px + qy = z$.

This eqn of the form ~~pp + qQ = R~~ $Pp + Qq = R$.

Here $P = x$, $Q = y$, $R = z$

The A.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(ie) $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$

Step:1

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\int \frac{dx}{x} = \int \frac{dy}{y} \Rightarrow \log x = \log y + \log c_1 \Rightarrow \log x - \log y = \log c_1$$

$\log\left(\frac{x}{y}\right) = \log c_1$

$\frac{x}{y} = c_1$

Step: 2

$\frac{dy}{y} = \frac{dz}{z}$

$\int \frac{dy}{y} = \int \frac{dz}{z}$

$\log y = \log z + \log c_2$

$\log y - \log z = \log c_2$

$\log(y/z) = \log c_2$

$y/z = c_2$

$\therefore$ The general soln $\phi(u, v) = 0$

(ie) $\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$.

(3) Find the soln of $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$.

Soln: Given $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

This eqn is of the form $Pp + Qq = R$

where $P = \sqrt{x}$, $Q = \sqrt{y}$ & $R = \sqrt{z}$.

Lagrange's subsidary eqn are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(ie) $\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$

Take $\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}}$

$\int \frac{dx}{\sqrt{x}} = \int \frac{dy}{\sqrt{y}}$

$2\sqrt{x} = 2\sqrt{y} + 2c_1$

$\sqrt{x} = \sqrt{y} + c_1$

$c_1 = \sqrt{x} - \sqrt{y}$

$u = \sqrt{x} - \sqrt{y}$

Take $\frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$

$\int \frac{dy}{\sqrt{y}} = \int \frac{dz}{\sqrt{z}}$

$2\sqrt{y} = 2\sqrt{z} + 2c_2$

$\sqrt{y} = \sqrt{z} + c_2$

$c_2 = \sqrt{y} - \sqrt{z}$

(ie) $v = \sqrt{y} - \sqrt{z}$

Hence the general soln is $f(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0$.

where f is arbitrary.

(ii) Method of multipliers:-

* choose the multipliers l, m, n (not necessarily constant) such that $lP+mQ+nR=0$.

Then $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$

$\Rightarrow \frac{l\,dx + m\,dy + n\,dz}{0} = \frac{dx}{P}$ $[\because lP+mQ+nR=0]$

$\Rightarrow l\,dx + m\,dy + n\,dz = 0.$

Now integrating we get $u = c_1$

* Now select another multipliers l', m', n' such that $l'P + m'Q + n'R = 0$.

Now $\frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R} = \frac{dz}{P}$

$\Rightarrow l'dx + m'dy + n'dz = 0.$

Solving this we get soln $v = c_2$

Now $u = c_1$ & $v = c_2$ constitute the required soln $\phi(u, v) = 0$.

(1) Solve $(y-z)p + (z-x)q = x-y$

Soln: This eqn is of the form $Pp + Qq = R$

Here $P = y-z$, $Q = z-x$, $R = x-y$

The A.E are $\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$

Step: 1 Choose the multipliers $(l, m, n) = (1, 1, 1)$

Consider $\dfrac{dx+dy+dz}{y-z+z-x+x-y} = \dfrac{dx}{x-y}$

$$\frac{d(x+y+z)}{0} = \frac{dx}{x-y}$$

$$\therefore \int d(x+y+z) = \int 0$$

$$\boxed{x+y+z = c_1}$$

Step: 2

Chose the multipliers $(l, m, n) = (x, y, z)$

$$\frac{x\,dx + y\,dy + z\,dz}{xy - xz + yz - xy + xz - yz} = \frac{dx}{y-z}$$

$$\int x\,dx + y\,dy + z\,dz = \int 0.$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2$$

$$\boxed{x^2 + y^2 + z^2 = c_2}$$

$\therefore$ The Soln is $\phi(x+y+z,\ x^2+y^2+z^2) = 0.$

⑤ Solve $(mz-ny)p+(nx-lz)q=ly-mx$ (+)

Soln: Given

$$(mz-ny)p+(nx-lz)q=ly-mx$$

This eqn is of the form $Pp+Qq=R$

where $P=mz-ny$, $Q=nx-lz$, $R=ly-mx$

The A.E is $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

(ie) $\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$

Using the multipliers x, y, z we get

$$\frac{xdx+ydy+zdz}{xmz-nxy+nxy-lzy+lyz-mxz}=\frac{dx}{mz-ny}$$

$$\frac{xdx+ydy+zdz}{0}=\frac{dx}{mz-ny}$$

$\Rightarrow xdx+ydy+zdz=0.$

Integrating $x^2+y^2+z^2=c_1$

Using the multipliers l, m, n we get

$$\frac{ldx+mdy+ndz}{0}=\frac{dx}{mz-ny}$$

$\Rightarrow ldx+mdy+ndz=0.$

Integrating, $lx+my+nz=0.$

$\therefore$ The General Soln is

$$\phi(x^2+y^2+z^2,\ lx+my+nz)=0.$$

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⑤. Solve $(3z-4y)p+(4x-2z)q = 2y-3x$.

Soln: Given $(3z-4y)p+(4x-2z)q = 2y-3x$.

This eqn is of the form $Pp+Qq=R$.

where $P = 3z-4y$, $Q = 4x-2z$, $R = 2y-3x$

The A.E are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(ie) $\frac{dx}{3z-4y} = \frac{dy}{4x-2z} = \frac{dz}{2y-3x} \longrightarrow ①$

Use lagrangian multipliers x, y, z we get

Each ratio $= \frac{xdx+ydy+zdz}{x(3z-4y)+y(4x-2z)+z(2y-3x)} = \frac{xdx+ydy+zdz}{0}$

(ie) $xdx+ydy+zdz = 0$.

Integrating we get

$\int xdx + \int ydy + \int zdz = 0$.

$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1}{2}$

$\Rightarrow \boxed{x^2+y^2+z^2 = c_1}$

Step 2 Using lagrangian multipliers 2, 3, 4 we get

Each ratio $= \frac{2dx+3dy+4dz}{6z-8y-12x-6z+8y-12x} = \frac{2dx+3dy+4dz}{0}$

(i.e) $2dx + 3dy + 4dz = 0$.

Integrating we get

$$\int 2dx + \int 3dy + \int 4dz = 0$$

$$2x + 3y + 4z = c_2$$

Hence the general soln $\phi(u,v) = 0$.

(i.e) $\phi(x^2+y^2+z^2,\ 2x+3y+4z) = 0$

where ϕ is arbitrary.

Soln of standard types of first order P.D. Eqns:-

The general form of a first order p.d.e is $f(x, y, z, p, q) = 0$,

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$

Types of solution:-

a) A soln in which the no. of arbitrary constants is equal to the no. of independent variables is called complete Integral (or) complete soln.

b) In complete integral if we give particular values to the arbitrary constant we get particular integral.

c) Singular integral:-

Let $f(x, y, z, p, q) = 0$ be a partial differential eqn whose complete integral is

$$\phi(x, y, z, a, b) = 0 \rightarrow (1)$$

Diff (1) p.w.r.to 'a' and 'b' then equal to zero we get

$$\frac{\partial \phi}{\partial a} = 0 \rightarrow (2)$$

$$\frac{\partial \phi}{\partial b} = 0 \rightarrow (3)$$

Eliminate a and b by using eqns (1) & (2) & (3).

The eliminant of a and b is called singular integral

Non-linear p.d.e

A partial differential eqn which involves first order partial derivatives p and q with degree higher than one and the products p and q is called non-linear P.D.E.

TYPE - I

$F(p, q) = 0$ (complete soln).

TYPE - II

$z = px + qy + f(p, q)$ [clairaut's form]

(singular soln).

TYPE - III

$F(z, p, q) = 0$ (complete soln).

TYPE - IV

$F_1(x, p) = F_2(y, q)$ (comp. soln.

TYPE - V

$F(x^m p, y^n q) = 0$ and $F(z, x^m p, y^n q) = 0$.

TYPE - VI

$F(z^m p, z^m q) = 0$ and $F_1(x, z^m p) = F_2(y, z^m q)$

Type-I form $F(p,q)=0$

To solve this type of problem we have to assume $z=ax+by+c$ be the soln of the given p.d.e.

* Here we have to find complete integral and singular integral.

note: But this type there is no singular integral.

① Solve $\sqrt{p}+\sqrt{q}=1$.

soln: Given $\sqrt{p}+\sqrt{q}=1 \longrightarrow ①$

This is of the form $F(p,q)=0$.

The trial soln is $z=ax+by+c$.

$\sqrt{p}+\sqrt{q}=1$

put $p=a, q=b$

where $\sqrt{a}+\sqrt{b}=1$

$\sqrt{b}=1-\sqrt{a}$

$b=(1-\sqrt{a})^2$

∴ The complete soln is $z=ax+(1-\sqrt{a})^2y+c$.

② Solve $p^2+q^2=4$.

soln: $p^2+q^2=4 \longrightarrow ①$

This is of the form $F(p,q)=0$.

The trial soln is $z=ax+by+c$.

where $a^2+b^2=4$.

$b^2=4-a^2$

$b=\pm\sqrt{4-a^2}$.

∴ The complete soln is $z=ax\pm(\sqrt{4-a^2})y+c$.

(3) ~~To find the general~~

Solve $p+q=pq$

Soln: This is of the form $F(p,q)=0$.

The trial soln is $z=ax+by+c$.

Complete integral:-

$z=ax+by+c$. where $a+b=ab$.

$b-ab=-a$

$b(1-a)=-a$

$b=\frac{-a}{1-a}$

$b=\frac{a}{a-1}$

$$\boxed{z=ax+\left(\frac{a}{a-1}\right)y+c}$$

Type: 2

Eqns of the form $F_1(x,p)=F_2(y,q)$

put $F_1(x,p)=F_2(y,q)=a$ (say)

Solving for p and q we get

$p=\phi_1(x,a)$ and $q=\phi_2(y,a)$

But $dz=pdx+qdy$

$dz=\phi_1(x,a)dx+\phi_2(y,a)dy$

Integrating on bothsides

$z=\int\phi_1(x,a)\,dx+\int\phi_2(y,a)\,dy+b$.

This eqn contains two arbitrary constants and hence it is the complete integral. The singular and general integrals are found out as usual.

① Solve $p^2+q^2 = x+y$

Soln: Given $p^2+q^2 = x+y$

$$p^2-x = y-q^2 = a \text{ (say)}$$

$p^2-x=a$ | $y-q^2=a$

$p^2=a+x$ | $q^2=y-a$

$p=\sqrt{a+x}$ | $q=\sqrt{y-a}$

from $dz = p\,dx + q\,dy$

$$dz = \sqrt{a+x}\,dx + \sqrt{y-a}\,dy.$$

$\int$ ing we get

$$z = \frac{(a+x)^{3/2}}{3/2} + \frac{(y-a)^{3/2}}{3/2} + b.$$

$$z = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(y-a)^{3/2} + b.$$

② Solve $p^2+q^2 = x^2+y^2$

Soln:

$$p^2-x^2 = y^2-q^2 = a^2 \text{ (say)}$$

$p^2-x^2=a^2$ | $y^2-q^2=a^2$

$p^2=a^2+x^2$ | $q^2=y^2-a^2$

$p=\sqrt{a^2+x^2}$ | $q=\sqrt{y^2-a^2}$

from $dz = p\,dx + q\,dy$

$$\int dz = \int\sqrt{a^2+x^2}\,dx + \int\sqrt{y^2-a^2}\,dy$$

$$z = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right) + \frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\log\left(y+\sqrt{y^2-a^2}\right) + b$$

is the required soln.

Type - III

Clairauts form:-

$$z = px + qy + f(p,q)$$

Its complete soln is $z = ax + by + f(a,b)$.

where $a = p$, $b = q$.

The singular integral is obtained as follows.

Consider $z = ax + by + f(a,b) \longrightarrow (1)$

D p w. r. to 'a' & 'b'

$x + \frac{\partial}{\partial a} f = 0 \rightarrow (2)$ & $y + \frac{\partial f}{\partial b} = 0 \rightarrow (3)$

Eliminating 'a' & b from (1) & (2) we get a singular integral.

Let $b = f(a)$ in (1)

D (1) p. w. r. to 'a' and eliminate the 'a' from last two eqns we get the general soln.

pbm

(1) Solve $z = px + qy + pq$

Soln: Given eqn of the form

$z = px + qy + f(p,q)$ (clairauts form)

put $p = a$, $q = b$.

The complete integral is $z = ax + by + ab \longrightarrow (1)$

To find singular integral:-

D (1) p. w. r. to 'a' & 'b' we get

$x + b = 0$	$y + a = 0$
$b = -x$	$a = -y$

Sub a & b values in ① we get

$$z = -yx - xy + xy$$

$xy + z = 0$ is the singular integral.

② Solve $z = px + qy + p^2 - q^2$

Soln: put $p = a$ & $q = b$

The Complete integral is $z = ax + by + a^2 - b^2 \rightarrow ①$

Singular integral:-

D. ① p.w.r.to 'a' & 'b' we get

$0 = x + 2a$	$0 = y - 2b$
$a = -x/2$	$b = y/2$

$$\therefore z = \left(\frac{x}{2}\right)x + \left(\frac{y}{2}\right)y + \left(-\frac{x}{2}\right)^2 - \left(y/2\right)^2$$

$$z = -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}.$$

$$z = -\frac{x^2}{4} + \frac{y^2}{4}$$

$\boxed{4z = y^2 - x^2}$ is the singular soln.

Type - iii

form $F(z, p, q) = 0$.

Let $u = x + ay$ and put $p = \frac{dz}{du}$ & $q = a\frac{dz}{du}$

in given eqn.

Then $\frac{dz}{du} = \phi(z, a)$ say

$$\int \frac{dz}{\phi(z,a)} = \int du$$

$$f(z,a) = u + c$$

$$\boxed{f(z,a) = x + ay + c}$$ which is the C.I.

pbm:

① Solve $p(1-q^2) = q(1-z)$

Soln: Given $p(1-q^2) = q(1-z) \longrightarrow ①$

This is of the form $f(z, p, q) = 0$.

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1 \quad \& \quad \frac{\partial u}{\partial y} = a$$

$\therefore p = \frac{dz}{du}$, $q = a\frac{dz}{du}$ in ① we get

$$\cancel{\frac{dz}{du}}\left[1 - \left(a\frac{dz}{du}\right)^2\right] = a\cancel{\frac{dz}{du}}(1-z)$$

$$1 - a^2\left(\frac{dz}{du}\right)^2 = a(1-z)$$

$$1 - a(1-z) = a^2\left(\frac{dz}{du}\right)^2$$

$$1 - a + az = a^2\left(\frac{dz}{du}\right)^2$$

$$\left(\frac{dz}{du}\right)^2 = \frac{1}{a^2}\left[1-a+az\right]$$

$$\frac{dz}{du} = \frac{1}{a}\sqrt{1-a+az}$$

$$dz = \frac{1}{a}\sqrt{1-a+az}\,du.$$

$$\int \frac{a}{\sqrt{1-a+az}}\,dz = \int du.$$

$$2\sqrt{1-a+az} = u+c.$$

Squaring on both sides

$$4(1-a+az) = (u+c)^2$$

$$4(1-a+az) = (x+ay+c)^2$$

which is the C.I.

(2) Solve. $z^2 = 1+p^2+q^2 \longrightarrow (1)$

Soln: Let $u = x+ay$

put $p = \frac{dz}{du}$ & $q = a\frac{dz}{du}$ in (1) we get

$$z^2 = 1+\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2$$

$$\left(\frac{dz}{du}\right)^2\left[1+a^2\right] = z^2-1$$

$$\left(\frac{dz}{du}\right)^2 = \frac{z^2-1}{1+a^2}$$

$$\frac{dz}{du} = \frac{\sqrt{z^2-1}}{\sqrt{1+a^2}}$$

$$\int \frac{dz}{\sqrt{z^2-1}} = \int \frac{du}{\sqrt{1+a^2}}.$$

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$$\cosh^{-1} z = \frac{1}{\sqrt{1+a^2}} u + c$$

$$= \frac{1}{\sqrt{1+a^2}} (x+ay) + c \quad \text{which is the C.I.}$$

Type - IV

① Solve $q = 2px$

Soln: Given $2px = q = a$ (say)

$2px = a$ | $q = a.$

$p = \frac{a}{2x}$

from $dz = p\,dx + q\,dy$

$$dz = \frac{a}{2x} dx + a\,dy$$

$$\int dz = \frac{a}{2} \int \frac{dx}{x} + a \int dy$$

$$z = \frac{a}{2} \log x + ay + c.$$

② Solve $\sqrt{p} + \sqrt{q} = x + y$

Soln: Given $\sqrt{p} - x = a$ | $y - \sqrt{q} = a$

$\sqrt{p} = a + x$ | $\sqrt{q} = y - a$

$p = (a+x)^2$ | $q = (y-a)^2$

from $dz = p\,dx + q\,dy$

$$\int dz = \int (a+x)^2 dx + \int (y-a)^2 dy.$$

$$z = \frac{(a+x)^3}{3} + \frac{(y-a)^3}{3} + c.$$

Eliminating 'a' from ② & ③ we get the general soln.

Linear pde's of second and higher order with constant coefficients:-

1. Homogeneous eqns.
2. Non-homogeneous eqns.

Defn:

A linear pde with constant coefficients in which all the partial derivatives are of the same order is called homogeneous; otherwise it is called non-homogeneous.

Homogeneous Linear Eqns with constant coefficients:-

The general form of n^{th} order P.D.E with constant coefficients is of the form

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + k_n \frac{\partial^n z}{\partial y^n} = F(x,y) \longrightarrow ①$$

put $\frac{\partial^r}{\partial x^r} = D^r$, $\frac{\partial^r}{\partial y^r} = D'^r$

$$\therefore ① \Rightarrow \left[D^n + k_1 D^{n-1} D' + \cdots + k_n D'^n\right] z = F(x,y)$$

ie) $f(D, D') z = F(x,y) \longrightarrow ②$

The general soln of ② $= C.F + P.I$

Procedure to find the C.F :-

Consider $\left[D^n + k_1 D^{n-1} D' + \dots + k_n D'^n\right] z = 0$

now put $D = m$ & $D' = 1$ we get the A.E as

$m^n + k_1 m^{n-1} + \dots + k_n = 0$.

Solving the above eqn we get the roots are

$m_1, m_2, m_3 \dots$

Then we write the C.F as follows

Roots of A.E	C.F
$m_1, m_2, m_3 \dots$ (Distinct roots)	$f_1(y+m_1x) + f_2(y+m_2x) + f_3(y+m_3x) + \dots$
m_1, m_1, m_3 (Two equal roots)	$f_1(y+m_1x) + xf_2(y+m_1x) + f_3(y+m_3x) + \dots$
m_1, m_2, m_3 (m) : 3-equal roots	$f_1(y+m_1x) + xf_2(y+m_1x) + x^2f_3(y+m_1x) + x^3f_4(y+m_1x)$

Rules to finding the P.I :-

$$P.I = \frac{1}{f(D,D')} F(x,y)$$

Case(i): when $F(x,y) = e^{ax+by}$

$$P.I = \frac{1}{f(D,D')} e^{ax+by}$$

put $D = a$ & $D' = b$

$$\Rightarrow P.I = \frac{1}{f(a,b)} e^{ax+by} \quad \text{if } f(a,b) \neq 0.$$

Suppose $f(a,b) = 0$.

$$P.I = \frac{x}{f'(D,D')} e^{ax+by}$$

$$= \frac{x}{f'(a,b)} e^{ax+by} \quad \text{if } f'(a,b) \neq 0$$

Suppose $f'(a,b) = 0$, then

$$P.I = \frac{x^2}{f''(D,D')} e^{ax+by}$$

$$P.I = \frac{x^2}{f''(a,b)} e^{ax+by} \quad \text{if } f''(a,b) \neq 0.$$

and so on.

Case (ii): when $f(x,y) = \sin(ax+by)$ (or) $\cos(ax+by)$.

$$P.I = \frac{1}{f(D^2, DD', D'^2)} \sin(ax+by) \text{ (or) } \cos(ax+by)$$

put $D^2 = -a^2$, $D'^2 = -b^2$, $DD' = -ab$

$$P.I = \frac{1}{f(-a^2,-ab,-b^2)} \sin(ax+by) \text{ (or) } \cos(ax+by)$$

if $f(-a^2,-ab,-b^2) \neq 0$

Suppose $Dr = 0$. then.

$$P.I = \frac{x}{f'(D^2, DD', D'^2)} \sin(ax+by) \text{ (or) } \cos(ax+by)$$

$$= \frac{x}{f'(-a^2, -ab, -b^2)} \sin(ax+by) \text{ (or) } \cos(ax+by)$$

Type (iii) when $F(x,y) = x^m y^n$

$$P.I = \frac{1}{f(D,D')} x^m y^n = f(D,D')^{-1} x^m y^n$$

Expand $f(D,D')^{-1}$ in ascending powers of $\frac{D'}{D}$ and operate on $x^m y^n$ term by term.

~~Case (iv)~~ Type (iv)

when $F(x,y) = \sin ax \sin by$

$$P.I = \frac{1}{f(D^2, D'^2)} \sin ax \sin by$$

$$= \frac{1}{f(-a^2, -b^2)} \sin ax \sin by$$

when $F(x,y) = \cos ax \cos by$

$$P.I = \frac{1}{f(D^2, D'^2)} \cos ax \cos by = \frac{1}{f(-a^2, -b^2)} \cos ax \cos by$$

Type (v):-

when $F(x,y) = e^{ax+by} \phi(x,y)$

$$P.I = \frac{1}{f(D,D')} e^{ax+by} \phi(x,y)$$

$$P.I = e^{ax+by} \frac{1}{f[D+a, D'+b]} \phi(x,y)$$

Type (vi):-

when $F(x,y) =$ Any function

$$P.I = \frac{1}{f(D,D')} F(x,y)$$

$C.F = \phi_1(y-3x) + \phi_2(y+\frac{1}{2}x)$

$\therefore z = \phi_1(y-3x) + \phi_2(2y+x)$

Type – I $\quad [R.H.S = e^{ax+by}]$

① Solve $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$

Soln: Given $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$

A.E is $m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0 \quad$ (if $D = m$, $D' = 1$).

$m = 2, 2$

$C.F = \phi_1(y+2x) + x\phi_2(y+2x)$

$P.I = \frac{1}{(D^2 - 4DD' + 4D'^2)} e^{2x+y}$

$= \frac{1}{2^2 - 4(2)(1) + 4(1)} e^{2x+y} \quad (D=2, D'=1)$

$= \frac{1}{4-8+4} e^{2x+y} = \frac{1}{0} e^{2x+y}$

$= \frac{x}{2D - 4D'} e^{2x+y}$

$= \frac{x}{2(2) - 4(1)} e^{2x+y}$

$= \frac{x}{0} e^{2x+y}$

$P.I = \frac{x^2}{2} e^{2x+y}$

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The soln is $z = C.F + P.I$

$$z = \phi_1(y+2x) + x\phi_2(y+2x) + \frac{x^2}{2} e^{2x+y}.$$

② Solve $(2D^2 - 5DD' + 2D'^2)z = e^{2x+y}$

Soln:

A.E is $2m^2 - 5m + 2 = 0.$

$2m^2 - 4m - m + 2 = 0.$

$2m(m-2) - (m-2) = 0.$

$(2m-1)(m-2) = 0$

$m = 1/2, \quad m = 2.$

$C.F = \phi_1(y + \frac{1}{2}x) + \phi_2(y+2x)$

$P.I = \frac{1}{2D^2 - 5DD' + 2D'^2} e^{2x+y}$

$D \to 2$

$D' \to 1$

$= \frac{1}{2(4) - 5(2)(1) + 2(1)} e^{2x+y}$

$= \frac{1}{8-10+2} e^{2x+y}$

$= \frac{x}{4D - 5D'} e^{2x+y}$

$= x \frac{1}{8-5} e^{2x+y}$

$P.I = \frac{x}{3} e^{2x+y}$

$\therefore$ The soln is $z = C.F + P.I$

$$z = \phi_1(y + \tfrac{1}{2}x) + \phi_2(y+2x) + \frac{x}{3} e^{2x+y}$$

③ find P.I of $(D^2-4DD')z = e^{3x+4y}$

Soln: $P.I = \frac{1}{D^2-4DD'} e^{3x+4y}$

$= \frac{1}{9-4(3)(4)} e^{3x+4y}$ | $D \to 3$, $D' \to 4$

$= \frac{1}{9-48} e^{3x+4y}$

$= \frac{-1}{39} e^{3x+4y}$

④ Find the P.I of $(D^2+2DD'+D'^2)z = e^{x-y}$

Soln: $P.I = \frac{1}{D^2+2DD'+D'^2} e^{x-y}$ | $D \to 1$, $D' \to -1$

$= \frac{1}{1+2(1)(-1)+1} e^{x-y}$

$= \frac{1}{1-2+1} e^{x-y}$

$= x. \frac{e^{x-y}}{2D+2D'} = x. \frac{1}{2(1)-2} e^{x-y}$

$P.I = \frac{x^2}{2} e^{x-y}$

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Type - IV [R.H.S = $\sin(ax+by)$ (or) $\cos(ax+by)$]

Formulae

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

① Find P.I $(D^2+4DD'-5D'^2)z = \sin(2x+3y)$

Soln:

$$P.I = \frac{1}{D^2+4DD'-5D'^2}\sin(2x+3y)$$

$D^2 = -4$

$D'^2 = -9$

$DD' = -(2)(3)$

$= -6$

$$= \frac{1}{-4+4(-6)-5(-9)}\sin(2x+3y)$$

$$= \frac{1}{-4-24+45}\sin(2x+3y)$$

$$= \frac{1}{45-28}\sin(2x+3y)$$

$$P.I = \frac{1}{17}\sin(2x+3y)$$

② Solve $(D^3-4D^2D'+4DD'^2)z = 6\sin(3x+6y)$.

Soln: A.E is $m^3-4m^2+4m=0$.

$m(m^2-4m+4)=0$.

$m=0$, $(m-2)^2=0$.

$m=0$, $m=2,2$

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$$C.F = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x).$$

1

$$P.I = \frac{1}{D^3 - 4D^2D' + 4DD'^2} 6\sin(3x+6y)$$

$$= 6 \cdot \frac{1}{D[D^2 - 4DD' + 4D'^2]} \sin(3x+6y).$$

$D^2 = -9$

$D'^2 = -36$

$DD' = -18$

$$= \frac{6}{D} \cdot \frac{1}{-9-4(-18)+4(-36)} \sin(3x+6y)$$

$$= \frac{6}{D} \cdot \frac{1}{-9+72-144} \sin(3x+6y).$$

$$= \frac{6}{D} \frac{1}{-81} \sin(3x+6y)$$

$$= -\frac{6}{81} \frac{1}{D}[\sin(3x+6y)]$$

$$= -\frac{2}{27} \int \sin(3x+6y)\,dx$$

$$= -\frac{2}{27}\left[\frac{-\cos(3x+6y)}{3}\right]$$

$$P.I = \frac{2}{81} \cos(3x+6y)$$

$\therefore$ the complete soln is

$$z = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) + \frac{2}{81}\cos(3x+6y)$$

⑮ Solve $(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y) + e^{3x+y}$

Soln:

A.E is $m^3 - 7m - 6 = 0$.

$m = -1,\ m = 3,\ m = -2$

$$\begin{array}{r|rrrr} & 1 & 0 & -7 & -6 \\ -1 & 0 & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & \boxed{0} \\ 3 & 0 & 3 & 6 & \\ \hline & 1 & 2 & \boxed{0} & \\ -2 & 0 & -2 & & \\ \hline & 1 & \boxed{0} & & \end{array}$$

$C.F = \phi_1(y+3x) + \phi_2(y-x) + \phi_3(y-2x)$

$$P.I_1 = \frac{1}{D^3 - 7DD'^2 - 6D'^3} \sin(x+2y)$$

$$\frac{1}{D(D^2) - 7DD'D' - 6D'^2D'}$$

$$= \frac{1}{D(-1) - 7(-2)D' - 6(-4)D'}$$

$$= \frac{1}{-D + 14D' + 24D'}$$

$$= \frac{1}{D^2D - 7DD'^2 - 6D'^2D'} \sin(x+2y)$$

$$= \frac{1}{D^2D - 7DD'D' - 6D'^2D'} \sin(x+2y)$$

$D^2 \to -1$

$D'^2 \to -4$

$DD' \to -2$

$$= \frac{\sin(x+2y)}{-D - 7(-2)D' - 6(-4)D'}$$

$$= \frac{\sin(x+2y)}{-D + 14D' + 24D'}$$

$$= \frac{\sin(x+2y)}{-D + 38D'}$$

$$= \frac{1}{-D + 38D'} \times \frac{D}{D} \sin(x+2y)$$

$$= \frac{D \sin(x+2y)}{-D^2 + 38DD'}$$

$$= \frac{\cos(x+2y)}{-(-1)+3(-2)}$$

$$= \frac{\cos(x+2y)}{1-26}$$

$$PI_1 = \frac{-1}{25}\cos(x+2y)$$

$$PI_2 = \frac{1}{D^3-7DD'^2-6D'^3} e^{3x+y}$$

$$= \frac{1}{27-7(3)(1)-6(1)} e^{3x+y}$$

$$= \frac{1}{27-21-6} e^{3x+y}$$

$$= \frac{1}{0} e^{3x+y}$$

$$= x \cdot \frac{1}{3D^2-7D'^2} e^{3x+y}$$

$$= x \cdot \frac{1}{3(9)-7(1)} e^{3x+y}$$

$$= x \cdot \frac{1}{27-7} e^{3x+y}$$

$$PI_2 = \frac{x}{20} e^{3x+y}$$

The soln is

$$z = C.F + P.I$$

$$z = \phi_1(y+3x)+\phi_2(y-x)+\phi_3(y-2x) - \frac{1}{25}\cos(x+2y)+\frac{x}{20}e^{3x+y}$$

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Type - (ii) [R.H.S $= x^r y^s$]

1) Solve $(D^2 - 7DD' + 6D'^2) z = xy$

Ans: Given $[D^2 - 7DD' + 6D'^2] z = xy$

The AE is $m^2 - 7m + 6 = 0$

$(m-6)(m-1) = 0$

$m = 6, m = 1$

$CF = \phi_1(y+x) + \phi_2(y+6x)$

$$PI = \frac{1}{D^2 - 7DD' + 6D'^2} xy$$

$$= \frac{1}{D^2\left[1 - \left(\frac{7D'}{D} - \frac{6D'^2}{D^2}\right)\right]} xy$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{7D'}{D} - \frac{6D'^2}{D^2}\right)\right]^{-1} xy$$

$$= \frac{1}{D^2}\left[1 + \frac{7D'}{D} - \frac{6D'^2}{D^2}\right] xy$$

$$= \frac{1}{D^2}\left[xy + \frac{7x}{D}\right] = \frac{1}{D^2}\left[xy + \frac{7x^2}{2}\right] = y\left[\frac{x^3}{6}\right] + \frac{7}{2}\left[\frac{x^4}{12}\right]$$

$$= \frac{x^3}{24}[4y + 7x]$$

Hence the complete integral is

$z = CF + PI$

$$z = \phi_1(y+x) + \phi_2(y+6x) + \frac{x^3}{24}(4y + 7x)$$

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② Solve $[D^2+3DD'+2D'^2]z = x+y$.

Soln: Given $(D^2+3DD'+2D'^2)z = x+y$

The A.E is $m^2+3m+2=0$.

$$m=-1,\ m=-2$$

$$C.F = \phi_1(y-x)+\phi_2(y-2x).$$

$$P.I = \frac{1}{D^2+3DD'+2D'^2}(x+y)$$

$$= \frac{1}{D^2}\left[1+\frac{3D'}{D}+\frac{2D'^2}{D^2}\right]^{-1}(x+y).$$

$$= \frac{1}{D^2}\left[1-\frac{3D'}{D}\right](x+y)$$

$$= \frac{1}{D^2}\left[x+y-\frac{3}{D}\right] = \frac{1}{D^2}\left[x+y-3x\right]$$

$$= \frac{1}{D^2}\left[y-2x\right] = \frac{yx^2}{2}-\frac{2x^3}{6}$$

$$= \frac{1}{6}(3yx^2-2x^3) = \frac{x^2}{6}(3y-2x)$$

∴ The complete Soln is $z = C.f + P.I$

$$z = \phi_1(y-x)+\phi_2(y-2x)+\frac{x^2}{6}(3y-2x)$$

③ Solve $(D^2+2DD'+D'^2)z = x^2y+e^{x-y}$

Soln: $(D^2+2DD'+D'^2)z = x^2y+e^{x-y}$.

The A.E is $m^2+2m+1=0$

$$(m+1)^2=0.$$

$$m=-1,-1$$

$$C.F = \phi_1(y-x)+x\phi_2(y-x)$$

$$P.I._1 = \frac{1}{(D+D')^2} x^2y$$

Use this method

$$P.I. = \frac{1}{D^2\left[1+\left(\frac{2DD'+D'^2}{D^2}\right)\right]} x^2y$$

$$= \frac{1}{D^2\left(1+\frac{D'}{D}\right)^2} x^2y = \frac{1}{D^2}\left[1+\frac{D'}{D}\right]^{-2} x^2y$$

$$= \frac{1}{D^2}\left[1-\frac{2D'}{D}+\frac{3D'^2}{D^2}\right](x^2y)$$

$$= \frac{1}{D^2}\left[x^2y - \frac{2}{D}(x^2)\right]$$

$$= \frac{1}{D^2}\left[x^2y - \frac{2x^3}{3}\right] = \frac{1}{D}\left[\frac{x^3y}{3} - \frac{2x^4}{12}\right]$$

$$P.I. = \frac{x^4y}{12} - \frac{x^5}{30}$$

$$z = C.F + P.I._1$$

$$= \phi_1(y-x) + x\phi_2(y-x) + \frac{x^4y}{12} - \frac{x^5}{30}$$

$$P.I._2 = \frac{1}{D^2+2DD'+D'^2} e^{x-y}$$

$$= \frac{1}{1-2+1} e^{x-y} \qquad D=1,\ D'=-1$$

$$= \frac{1}{0} e^{x-y}$$

$$= x \frac{1}{2D+2D'} e^{x-y}$$

$$= \frac{x}{2} \frac{1}{D+D'} e^{x-y}$$

$$= \frac{x}{2} \frac{1}{1-1} e^{x-y}$$

$$= \frac{x}{2} \frac{1}{0} e^{x-y}$$

$= \left(\frac{x}{2}\right) x \frac{1}{1} e^{x-y}$

$= \frac{x^2}{2} e^{x-y}$

$\therefore z = C.F + P.I_1 + P.I_2$

$$z = \phi_1(y-x) + x\phi_2(y-x) + \frac{x^4 y}{12} - \frac{x^5}{30} + \frac{x^2}{2} e^{x-y}$$

Type - IV $\left[R.H.S = e^{ax+by} \phi(x,y) \right]$

① Solve $[D^2 - 2DD' + D'^2]z = x^2 y^2 e^{x+y}$

Sol: Given $[D^2 - 2DD' + D'^2]z = x^2 y^2 e^{x+y}$

The A.E is $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1.$

$C.F = \phi_1(y+x) + x\phi_2(y+x)$

$$P.I = \frac{1}{(D-D')^2} x^2 y^2 e^{x+y}$$

$$= e^{x+y} \frac{1}{[(D+1)-(D'+1)]^2} x^2 y^2$$

$$= e^{x+y} \frac{1}{(D-D')^2} x^2 y^2$$

use this

$$P.I = \frac{1}{D^2\left[1 - \left(\frac{2DD' - D'^2}{D^2}\right)\right]} x^2 y^2$$

$$= e^{x+y} \frac{1}{D^2}\left[1 - \frac{D'}{D}\right]^{-2} x^2 y^2$$

$$= e^{x+y} \frac{1}{D^2}\left[1 + 2\frac{D'}{D} + \frac{3D'^2}{D^2}\right] x^2 y^2$$

$$= e^{x+y} \frac{1}{D^2}\left[x^2 y^2 + \frac{2}{D}(2x^2 y) + \frac{3}{D^2}[2x^2]\right]$$

$$= e^{x+y}\left[y^2\frac{1}{D^2}(x^2)+4y\frac{1}{D^3}(x^2)+6\frac{1}{D^4}(x^2)\right]$$

$$= e^{x+y}\left[\frac{1}{12}x^4y^2+\frac{1}{15}x^5y+\frac{1}{60}x^6\right].$$

The general soln is

$$z = \phi_1(y+x)+x\phi_2(y+x)+\left(\frac{1}{12}y^2+\frac{1}{15}xy+\frac{1}{60}x^2\right)x^4e^{x+y}$$

(9) Solve $(D^3+D^2D'-DD'^2-D'^3)z = e^x\cos 2y$.

Soln: Given $(D^3+D^2D'-DD'^2-D'^3)z = e^x\cos 2y$

The AE is $m^3+m^2-m-1=0$

$(m+1)^2(m-1)=0$

$m = 1, -1, -1$

$$\begin{array}{r|rrrr} & 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 2 & 1 \\ \hline & 1 & 2 & 1 & 0 \end{array}$$

$$C.F = \phi_1(y+x)+\phi_2(y-x)+x\phi_3(y-x)$$

$$P.I = \frac{1}{D^3+D^2D'-DD'^2-D'^3}e^x\cos 2y.$$

$$= e^x\frac{\cos 2y}{(D+1)^3+(D+1)^2D'-(D+1)D'^2-D'^3}$$

$D=0$, $D'=2i$

$$= e^x \text{ R.P of } \frac{e^{i2y}}{(D+1)^3+(D+1)^2D'-(D+1)D'^2-D'^3}$$

$$= e^x \text{ R.P of } \frac{e^{i2y}}{1+2i+4+8i}$$

$5+10i$

$1+2i$

$\times\frac{1-2i}{1-2i}$

$1-2i$

$$= \frac{e^x}{5}\text{ R.P of }\frac{1-2i}{(1+2i)(1-2i)}e^{i2y}.$$

$$= \frac{1}{5}e^x\frac{1}{5}\text{ R.P of }(1-2i)\left[\cos 2y+i\sin 2y\right]$$

$$= \frac{e^{x}}{25}\left[\cos 2y + 2\sin 2y\right]$$

The Complete Soln is $z = C.F + P.I$

$$z = \phi_1(y+x) + \phi_2(y-x) + x\phi_2(y-x) + \frac{e^{x}}{25}\left[\cos 2y + 2\sin 2y\right]$$

Type - VI [R.H.S $= y\cos x$]

formulae:

$$\frac{1}{D-mD'} f(x,y) = \int F(x, c-mx)\,dx \quad \text{where } y = c-mx$$

$$\frac{1}{D+mD'} f(x,y) = \int F(x, c+mx)\,dx \quad \text{where } y = c+mx.$$

① Solve $(D^2+DD'-6D'^2)z = y\cos x$.

Ans: Given $(D^2+DD'-6D'^2)z = y\cos x$.

The AE is $m^2+m-6=0$.

$m = 2, -3$.

$$C.F = \phi_1(y+2x) + \phi_2(y-3x).$$

$$PI = \frac{1}{D^2+DD'-6D'^2}\, y\cos x$$

$$= \frac{1}{(D-2D')(D+3D')}\, y\cos x$$

$$= \frac{1}{(D-2D')}\int (c+3x)\cos x\,dx \quad \text{when } y = c+3x.$$

$$= \frac{1}{D-2D'}\left[(c+3x)\sin x - 3\int \sin x\,dx\right] \quad \text{when } c = y-3x.$$

$$= \frac{1}{D-2D'}\left[y\sin x + 3\cos x\right]$$

$$= \int\left[(c-2x)\sin x + 3\cos x\right]dx \quad \text{when } y = c-2x.$$

$$= \left[(c-2x)(-\cos x) - (-2)(-\sin x) + 3\sin x\right) \quad \text{when}$$

$$c = y+2x.$$

$$= -y\cos x + \sin x$$

$$= \frac{e^{x} \cos(y)}{-\cos} = \frac{e^{x}}{-1}$$

① $(D^2 + DD' - 6D'^2)z = y\cos x$.

$m = 2, -3$.

$C.F = \phi_1(y+2x) + \phi_2(y-3x)$.

$$P.I = \frac{1}{D^2 + DD' - 6D'^2} y\cos x$$

$$= \frac{1}{D^2\left(1 + \left(\frac{DD' - 6D'^2}{D^2}\right)\right)} y\cos x = \frac{1}{D^2\left[1 + \left(\frac{D'}{D} - \frac{6D'^2}{D^2}\right)\right]} y\cos x$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{D'}{D} - \frac{6D'^2}{D^2}\right)\right]^{-1} (y\cos x)$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{D'}{D}\right)\right](y\cos x) = \frac{1}{D^2}\left[y\cos x - \frac{1}{D}(\cos x)\right]$$

$$= \frac{1}{D^2}\left[y\cos x - \sin x\right]$$

$$= \frac{1}{D}\left(y\sin x + \cos x\right) = -y\cos x + \sin x$$

$P.I = \sin x - y\cos x$

$z = C.F + P.I$.

② Solve $(D^2 - 5DD' + 6D'^2)z = y\sin x$.

$m = 2, 3$

$C.F = \phi_1(y+2x) + \phi_2(y+3x)$.

$$P.I = \frac{1}{D^2 - 5DD' + 6D'^2} y\sin x$$

$$= \frac{1}{D^2\left[1 - \left(\frac{5DD' - 6D'^2}{D^2}\right)\right]} (y\sin x)$$

$$= \frac{1}{D^2\left[1 - \left(\frac{5D'}{D} - \frac{6D'^2}{D^2}\right)\right]} (y\sin x)$$

$$= \frac{1}{D^2}\left[1 - \frac{5D'}{D}\right]^{-1} (y\sin x)$$

UNIT - IV

Applications of partial Differential Equations

Boundary value problems:-

Defn: Boundary conditions:-

The conditions which are given from time $t=0$ are called initial conditions. The conditions at the boundary of the region are called boundary conditions.

Defn: The partial differential eqns commonly arise from the mathematical formulation of physical problems. We have to find solns of such eqns which satisfy certain initial and boundary conditions. Such problems are called Boundary value problems.

Method of seperation of variables:-

Let u be the dependent variable and x, y are independent variables in the given partial diff eqn, then the solution is

$$u(x,y) = X(x) \cdot Y(y)$$

where $X(x) \Rightarrow$ a function of x alone.

$Y(y) \Rightarrow$ a function of y alone.

After substituting $u(x,y)$, the p.d.e is converted into two ordinary diff. eqns.

Classification of P.D.Eqns of Second order:-

The general second order linear partial differential eqn is of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G.$$

which can be written as

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G.$$

where A, B, C, D, E, F, G are constants (or) functions of x and y only.

i) The above eqn is elliptic if $B^2-4AC < 0$.

(ii) The above eqn is parabolic if $B^2-4AC = 0$.

(iii) The above eqn is Hyperbolic if $B^2-4AC > 0$.

Ex:1

Classify the following eqns.

(or)

Find the nature of the P.D.E

i) $u_{xx} + xu_{yy} = 0$

The given eqn is of the form $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$

Here $A = 1,\ B = 0,\ C = x$

$B^2 - 4AC = 0 - 4x = -4x$

$B^2 - 4AC < 0$ for $x > 0$

$B^2 - 4AC > 0$ for $x < 0$

$\therefore$ The given eqn is elliptic for $x > 0$

$\therefore$ The given eqn is hyperbolic for $x < 0$

② $4u_{xx} + 4u_{xy} + u_{yy} + 2u_x - u_y = 0$.

Soln: Here $A = 4$; $B = 4$, $C = 1$.

$B^2 - 4AC = 16 - 4(4)(1) = 16 - 16 = 0$.

$\therefore B^2 - 4AC = 0$.

The given eqn is parabolic eqn.

③ $u_{xx} - y^4 u_{yy} = 2y^3 u_y$.

Soln: Here $A = 1$, $B = 0$, $C = -y^4$.

$B^2 - 4AC = 0 - 4(1)(-y^4) = 4y^4$.

If $y = 0$ we get $B^2 - 4AC = 0$ (parabolic eqn).

If $y > 0$ (or) $y < 0$ we get $B^2 - 4AC > 0$ (Hyperbolic eqn).

$\therefore$ The eqn is hyperbolic eqn.

④ classify the p.d.e. $(1+x)^2 u_{xx} - 4x u_{xy} + u_{yy} = x$

Soln: Here $A = (1+x)^2$

$B = -4x$

$C = 1$

$B^2 - 4AC = 16x^2 - 4(1+x)^2$

$= 16x^2 - 4 - 8x - 4x^2$

$= 12x^2 - 8x - 4$

$B^2 - 4AC = 4[3x^2 - 2x - 1]$

Initial velocity = 0	Initial velocity ≠ 0
1. $y=0$ at $x=0$	1. $y=0$ at $x=0$
2. $y=0$ at $x=l$	2. $y=0$ at $x=l$
3. $\frac{\partial y}{\partial t}=0$ at $t=0$	3. $y=0$ at $t=0$
4. $y=f(x)$ at $t=0$	4. $\frac{\partial y}{\partial t}=g(x)$ at $t=0$.

Solutions of one Dimensional wave eqn:-

Consider a tightly stretched elastic string of length l with fixed ends A and B.

Let the string be given a small displacement $\perp r$ to its length.

Then the displacement y of a point P of the string at a distance x from A at any time t is given by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$ where c is a constant.

This eqn gives the transverse vibrations of the string and is called the one dimensional wave eqn.

three different solutions of wave eqns are

i) $y(x,t) = (c_1 e^{-px} + c_2 e^{px})(c_3 e^{-pat} + c_4 e^{pat})$

(ii) $y(x,t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos pat + c_8 \sin pat)$

(iii) $y(x,t) = (c_9 x + c_{10})(c_{11} t + c_{12})$.

Note: In pbms of vibration of string, the correct solution (or) Proper solution i.e.,

$$y(x,t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat)$$

Defn: one dimensional wave eqn

The eqn of transverse vibrations of the string is called the one dimensional wave eqn and is given by,

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

where $a^2 = \frac{T}{m}$ T – Tension

m – mass per unit length.

Type – I (elastic) (Zero Initial velocity) (No Initial velocity)

① A tightly stretched string is fixed, at the end points $x=0$ and $x=l$ is initially in a position given by $y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest from this position; find the displacement 'y' at any time and at any distance from the end $x=0$.

Soln:

The displacement $y(x,t)$ in the string at any time 't' satisfies ODWE $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ → ①

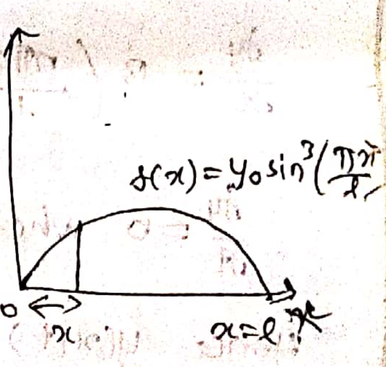
The boundary conditions are

i) $y(0,t)=0$ for all $t \geq 0$

(ii) $y(l,t)=0$ for all $t \geq 0$

(iii) $\frac{\partial y(x,0)}{\partial t}=0$ when $t=0$ &

(iv) $y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$, $0<x<l$



The suitable Soln of $y(x,t)$ can be taken as

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \rightarrow ②$$

By (i) $y(0,t)=0$

$\Rightarrow A(C\cos pat + D\sin pat) = 0$

$\therefore \boxed{A=0}$ Here $(C\cos pat + D\sin pat \neq 0)$

$\therefore y(x,t) = B\sin px\,(C\cos pat + D\sin pat) \rightarrow (2)$

By (ii) $y(l,t) = 0$, for all t in (3).

$\Rightarrow B\sin pl\,[C\cos pat + D\sin pat] = 0$

$B \neq 0$, $\sin pl = 0 \Rightarrow$

$\sin pl = \sin n\pi$

$pl = n\pi$

$$\boxed{p = \frac{n\pi}{l}}$$

$\therefore$ put $p = \frac{n\pi}{l}$ in (3) we get

$$y(x,t) = B\sin\frac{n\pi x}{l}\left[C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right] \rightarrow (4)$$

$$\frac{\partial y}{\partial t} = B\left(\frac{n\pi a}{l}\right)\sin\frac{n\pi x}{l}\left[-C\sin\frac{n\pi at}{l} + D\cos\frac{n\pi at}{l}\right]$$

$\frac{\partial y}{\partial t} = 0$ when $t = 0 \Rightarrow \boxed{D = 0}$

Hence $y(x,t) = B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$, when $B_n = BC$

The most general soln is given by

$$y(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \rightarrow (5)$$

Using condition $y(x,0) = y_0\left[\sin^3\frac{\pi x}{l}\right]$ in (5)

we have $\sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l} = y_0\sin^3\frac{\pi x}{l}$

$$\Rightarrow B_1 \sin\frac{\pi x}{l} + B_2 \sin\frac{2\pi x}{l} + B_3 \sin\frac{3\pi x}{l} + \cdots$$

$$= \frac{y_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{2\pi x}{l}\right]$$

$\therefore B_1 = \frac{3y_0}{4}$, $B_3 = -\frac{y_0}{4}$ and $B_2 = B_4 = B_5 = \cdots = 0$.

$\sin^3\theta = \frac{1}{4}[3\sin\theta - \sin 3\theta]$

The required soln is given by

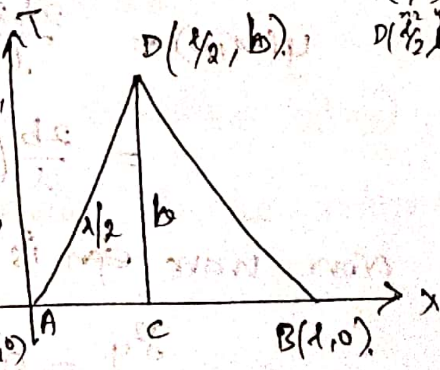
$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l}$$

(ie) $y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi a t}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi a t}{l}$

(2) A ~~taut~~ ~~tightly~~ tightly stretched string of length l is fastened at ends. $x=0$ & $x=l$. The mid point of the string is taken to a height of 'b' h and then released from rest in that position. Find the displacement of the string at any time.

Soln: Let the length of the string l.

The eqn of the string on line AD is



$A(0,0)$ $D(l/2, b)$

$$\frac{x-0}{0-l/2} = \frac{y-0}{0-b} \qquad \frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-0}{b-0} = \frac{x-0}{l/2-0}$$

$$\frac{x}{-l/2} = \frac{y}{-b} \qquad \frac{y}{b} = \frac{x}{l/2}$$

$$-bx = -\frac{ly}{2} \qquad y = \frac{2bx}{l} \quad (0,0)$$

$$y = \frac{2bx}{l}, \quad 0 < x < l/2.$$

$\therefore$ Eqn of the string in the interval $(0, l/2)$ is $\frac{2bx}{l}$.

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The eqn of the string DB is

$$x_1 \quad y_1$$
$$D(l/2, b)$$
$$x_2 \quad y_2$$
$$B(l, 0)$$

$$\frac{x - l/2}{l/2 - l} = \frac{y-b}{b-0}$$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-b}{0-b} = \frac{x-l/2}{l-l/2}$$

$$\frac{y-b}{-b} = \frac{2x-l}{2} \times \frac{2}{l}$$

$$y-b = -\frac{b}{l}(2x-l)$$

$$b(x - l/2) = -\frac{l}{2}(y-b)$$

$$y-b = -\frac{2b}{l}\left(\frac{2x-l}{2}\right)$$

$$y-b = \frac{-2bx+bl}{l}$$

$$\Rightarrow y = \frac{-2bx+bl}{l} + b = \frac{-2bx+bl+bl}{l}$$

$$y = \frac{2b}{l}[l-x].$$

∴ The eqn of the string in the interval $(l/2, l)$ is

$$y = \frac{2b}{l}(l-x).$$

Hence initially the displacement of the string is in the form

$$y(x,0) = \frac{2bx}{l}, \quad 0 < x < l/2$$

$$= \frac{2b}{l}(l-x), \quad \frac{l}{2} < x < l.$$

Now wave eqn is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \longrightarrow (1)$$

The boundary conditions are

i) $y(0,t) = 0$ for all $t \geq 0$

(ii) $y(l,t) = 0$ for all $t \geq 0$

(iii) $\frac{\partial y}{\partial t}(x,0) = 0 \quad \forall x$ in $(0,l)$.

(iv) $y(x,0) = \frac{2bx}{l}, \quad 0<x<l/2$

$= \frac{2b}{l}(l-x), \quad \frac{l}{2}<x<l$.

The soln of wave eqn ① satisfying the boundary conditions (i), (ii) and (iii) is

$$y(x,t) = B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \qquad \text{(Refer eqn ①)}$$

The most general soln is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \longrightarrow ②$$

Applying B.C (iv) in ② we get

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = f(x) \text{ (say)}. \longrightarrow ③$$

where $f(x) = \frac{2bx}{l}, \quad 0<x<l/2$

$= \frac{2b}{l}(l-x), \quad \frac{l}{2}<x<l$.

Now to find B_n expand the function $f(x)$ a half range fourier sine series in the interval $0<x<l$.

$$\text{Then } f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \longrightarrow ④$$

$$\text{where } b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

from ③ & ④ we get $B_n = b_n$.

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$$\therefore B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} \frac{2bx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}\frac{2b}{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{4b}{l^2}\left[\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{4b}{l^2}\left[\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right) - 1\left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right)\right]_0^{l/2} + \left[(l-x)\left(\frac{-\cos n\pi x/l}{n\pi/l}\right) + \left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right)\right]_{l/2}^{l}\right]$$

$$= \frac{4b}{l^2}\left[\cancel{\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2}} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \cancel{\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{4b}{l^2}\left[\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] \longrightarrow (5)$$

Sub (5) in (2) we get

$$y(x,t) = \sum_{n=1}^{\infty}\frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$ //

(or) $$y(x,t) = \frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\sin\left(\frac{(2n-1)\pi x}{l}\right)\cos\left(\frac{(2n-1)\pi at}{l}\right)$$

Type-II [Vibrating String with Non-zero initial velocity]:

(Initial velocity $\neq 0$)

1. $y=0$ at $x=0$
2. $y=0$ at $x=l$
3. $y=0$ at $t=0$
4. $\frac{\partial y}{\partial t} = g(x)$ at $t=0$.

1. $y(0,t)=0$ for all $t>0$
2. $y(l,t)=0$ for all $t>0$
3. $y(x,0)=0$ when $t=0$.
4. $\frac{\partial y}{\partial t}(x,0) = g(x)$

pbms:

1. If a string of length l is initially at rest in its equilibrium position and each of its points is given the velocity $V_0 \sin^3 \frac{\pi x}{l}$, $0<x<l$, determine the displacement of a point distant x from one end at time t.

Soln:

The wave eqn is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \longrightarrow (1)$.

The boundary conditions are

(i) $y(0,t)=0$, for all $t>0$

(ii) $y(l,t)=0$, for all $t>0$

(iii) $y(x,0)=0$, $0<x<l$.

(iv) $\frac{\partial y}{\partial t}(x,0) = V_0 \sin^3 \frac{\pi x}{l}$, $0<x<l$

$\therefore$ The suitable soln is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \rightarrow (2)$$

By (i) in (2) we get

$$A=0$$

$$y(x,t) = B\sin px\,(C\cos pat + D\sin pat) \rightarrow (3).$$

By (ii) in (3) we get

$$\boxed{p = \frac{n\pi}{l}}$$

$$\therefore y(x,t) = B\sin\left(\frac{n\pi x}{l}\right)\left[C\cos\left(\frac{n\pi at}{l}\right) + D\sin\left(\frac{n\pi at}{l}\right)\right] \rightarrow (4)$$

By (iii) in (4) we get

$$0 = y(x,0) = B\sin\left(\frac{n\pi x}{l}\right)\left[C\cos(0) + D\sin(0)\right]$$

$$0 = B\sin\left(\frac{n\pi x}{l}\right) C$$

Here $B \neq 0$.

$$\Rightarrow \boxed{C=0}$$

$\therefore y(x,t) = BD \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi at}{l}\right)$

$y(x,t) = B_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi at}{l}\right)$

where $BD = B_n$.

$\therefore$ The most general soln is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi at}{l}\right)$$

$$\frac{dy}{dt} = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right)\left(\frac{n\pi a}{l}\right)$$

Apply (iv) in (5) we get

$$\frac{dy}{dt}(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)\left(\frac{n\pi a}{l}\right)$$

$$V_0 \sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right) \text{ where } C_n = B_n\left(\frac{n\pi a}{l}\right)$$

$$C_1 \sin\left(\frac{\pi x}{l}\right) + C_2 \sin\left(\frac{2\pi x}{l}\right) + C_3 \sin\left(\frac{3\pi x}{l}\right) + \cdots$$

$$= V_0\left[\frac{3\sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)}{4}\right]$$

Equating the coefficient on both sides

$$C_1 = \frac{3V_0}{4}\ ;\ C_3 = \frac{-V_0}{4}\ ,\ C_2 = C_4 = C_5 = \cdots = 0.$$

$$C_n = B_n\left(\frac{n\pi a}{l}\right)$$

$$\Rightarrow B_n = \frac{l C_n}{n\pi a}$$

put $n=1$, $B_1 = \frac{lC_1}{\pi a} = \frac{l\left(\frac{3V_0}{4}\right)}{\pi a} = \frac{3lV_0}{4\pi a}$

Put n=3,

$$B_3 = \frac{l c_3}{3\pi a} = \frac{l\left(-\frac{v_0}{4}\right)}{3\pi a} = \frac{-l v_0}{12\pi a}$$

∴ the general soln is

$$y(x,t) = B_1 \sin\left(\frac{\pi x}{l}\right) \sin\left(\frac{\pi a t}{l}\right) + B_3 \sin\left(\frac{3\pi x}{l}\right) \sin\left(\frac{3\pi a t}{l}\right)$$

$$\boxed{\therefore y(x,t) = \frac{3 l v_0}{4\pi a} \sin\left(\frac{\pi x}{l}\right) \sin\left(\frac{\pi a t}{l}\right) - \frac{l v_0}{12\pi a} \sin\left(\frac{3\pi x}{l}\right) \sin\left(\frac{3\pi a t}{l}\right)}$$

(2) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating string giving each point a velocity $\lambda x(l-x)$ show that displacement

$$y(x,t) = \frac{8\lambda l^3}{a\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin\frac{(2n-1)\pi x}{l} \cdot \sin\frac{(2n-1)\pi a t}{l}$$

Soln: The wave eqn is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$.

The boundary conditions are

i) $y(0,t)=0$, for all $t>0$.

(ii) $y(l,t)=0$, for all $t>0$.

(iii) $y(x,0)=0$, $0<x<l$.

(iv) $\frac{\partial y}{\partial t}(x,0) = \lambda x(l-x)$, $0<x<l$.

The Suitable soln is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \rightarrow ②$$

from eqn's ① to ⑤ is same as previous pbms.

we get

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi at}{l}\right) \rightarrow ⑤$$

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right)\left(\frac{n\pi a}{l}\right)$$

$$\lambda x(l-x) = \frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)\left(\frac{n\pi a}{l}\right)$$

$$\therefore \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right) = \lambda x(l-x) \rightarrow ⑥$$

where $C_n = B_n\left(\frac{n\pi a}{l}\right)$.

To find C_n:-

Expand $\lambda x(l-x)$ in a half range fourier sine series in $(0,l)$.

$$f(x) = \lambda x(l-x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \rightarrow ⑦$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

from ⑥ & ⑦ we get

$$\boxed{b_n = C_n}$$

$$\therefore C_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

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$$= \frac{2}{l}\int_0^l \lambda x(l-x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2\lambda}{l}\int_0^l (lx-x^2)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2\lambda}{l}\left\{\left[(lx-x^2)\left[\frac{-\cos(n\pi x/l)}{n\pi/l}\right]-(l-2x)\left[\frac{-\sin(\frac{n\pi x}{l})}{n^2\pi^2/l^2}\right]\right.\right.$$

$$\left.\left. +(-2)\left[\frac{\cos(n\pi x/l)}{n^3\pi^3/l^3}\right]\right]\right\}_0^l$$

$$= \frac{2\lambda}{l}\left[0-0-2\frac{\cos n\pi}{\frac{n^3\pi^3}{l^3}}-0-0+2\frac{\cos 0}{n^3\pi^3/l^3}\right]$$

$$= \frac{2\lambda}{l}\left[\frac{-2(-1)^n+2}{n^3\pi^3/l^3}\right]$$

$$= \frac{2\lambda}{l}\cdot\frac{2l^3}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$C_n = \frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$B_n\left(\frac{n\pi a}{l}\right) = \frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$B_n = \frac{4\lambda l^2}{n^3\pi^3}\cdot\frac{l}{n\pi a}\left[1-(-1)^n\right]$$

$$\therefore B_n = \frac{4\lambda l^3}{a n^4\pi^4}\left[1-(-1)^n\right]$$

$$B_n = \begin{cases} 0 , & \text{if } n \text{ is even} \\ \dfrac{8\lambda l^3}{a n^4 \pi^4} , & \text{if } n \text{ is odd} \end{cases}$$

Sub. B_n value in (5) we get

$$y(x,t) = \sum_{n=odd}^{\infty} \frac{8\lambda l^3}{a n^4 \pi^4} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi a t}{l}\right)$$

$$y(x,t) = \frac{8\lambda l^3}{a\pi^4} \sum_{n=odd}^{\infty} \frac{1}{n^4} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi a t}{l}\right)$$

$$y(x,t) = \frac{8\lambda l^3}{a\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin\left(\frac{(2n-1)\pi x}{l}\right) \sin\frac{(2n-1)\pi a t}{l}$$

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ONE DIMENSIONAL HEAT EQUATION:-

The one dimensional heat eqn is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } \alpha^2 = \frac{K}{c\rho}$$

ρ - density
c - Specific heat
K - thermal conductivity of the material.

Three possible solns of the one dimensional heat eqn are.

i) $u(x,t) = (A_1 e^{px} + A_2 e^{-px})(A_3 e^{c^2p^2t})$

(ii) $u(x,t) = (A_4 \cos px + A_5 \sin px)(A_6 e^{-c^2p^2t})$

(iii) $u(x,t) = (A_7 x + A_8) A_9$

The boundary conditions are:-

i) $u(0,t) = k_1°c$ for all $t \geq 0$.

(ii) $u(l,t) = k_2°c$ for all $t \geq 0$

(iii) $u(x,0) = f(x)$, $0 < x < l$.

The suitable soln of the ODHE is

$$u(x,t) = (A\cos px + B\sin px)\, e^{-c^2p^2t}.$$

STEADY STATE SOLUTION OF TWO DIMENSIONAL HEAT EQUATION: (INSULATED EDGES EXCLUDED)

TWO DIMENSIONAL HEAT FLOW EQUATIONS

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

ie) $\nabla^2 u = 0$

which is known as Laplace's eqn. in two-dimensions;

The possible solns of the Laplace eqns are

i) $u(x,y) = (A_1 e^{px} + A_2 e^{-px})(A_3 \cos py + A_4 \sin py)$

ii) $u(x,y) = (A_5 \cos px + A_6 \sin px)(A_7 e^{py} + A_8 e^{-py})$

iii) $u(x,y) = (A_9 x + A_{10})(A_{11} y + A_{12})$.

Q) What is the basic difference between the solns of ODWE and ODHE.

Soln: Soln of the ODWE is of periodic in nature. But Soln of the ODHE is not of periodic in nature.

In steady state conditions derive the soln of one dimensional heat flow eqn.

Soln: The ODHE is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow (1)$

When the steady state conditions exist the heat flow eqn is independent of time t.

$$\therefore \frac{\partial u}{\partial t} = 0$$

$\therefore$ ① $\Rightarrow$ the heat flow eqn becomes

$$\frac{\partial^2 u}{\partial x^2} = 0 \quad , \text{ Integrating twice we get } u(x) = c_1 x + c_2 .$$

$\therefore$ The soln is $u(x) = c_1 x + c_2$.

(or)

pbms: $\quad u(x) = \left(\frac{b-a}{l}\right)x + a$.

① A rod 30 cm long has its ends A and B kept at $20°c$ and $80°c$ respectively until steady state conditions prevail. find the steady state temperature in the rod. $\quad u = \left(\frac{b-a}{l}\right)x + a$.

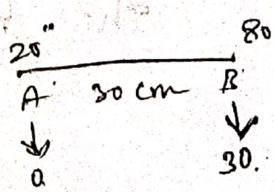
Soln: $\quad$ Let $l = 30$ cm

when steady state condition prevail the heat flow eqn, is

$$\frac{\partial^2 u}{\partial x^2} = 0 \quad \text{ie) } u(x) = c_1 x + c_2 \longrightarrow ①$$

when steady state conditions exists,

$\therefore$ the boundary conditions are



$u(0) = 20, \quad u(l) = 80.$

① $\Rightarrow$ Apply $u(0) = 20$ in ① we get

$u(0) = c_1(0) + c_2 = 20$

$$\Rightarrow \boxed{c_2 = 20}$$

Apply $u(l)=80$ in ① we get

① $\Rightarrow$ $u(l) = c_1(l) + c_2 = 80$

$c_1(30)+20=80$

$30c_1 = 80-20$

$c_1 = \frac{60}{30}$

$\boxed{c_1=2}$

Sub c_1 & c_2 values in ① we get

$\boxed{u(x) = 2x+20}$

② An insulated rod of length $l=60$ cm has its ends at A and B maintained at $30°c$ and $40°c$ respectively. Find the steady state solution.

Soln: The heat flow eqn is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

when steady state condition exist, the heat flow eqn becomes $\frac{\partial^2 u}{\partial x^2}=0$.

ie) $u(x) = c_1x + c_2 \longrightarrow$ ①

The boundary conditions are

$u(0)=30$, $u(l)=40$, $l=60$ cm

Apply $u(0)=30$ in ① we get

$u(0) = c_1(0)+c_2 = 30$

$\Rightarrow \boxed{c_2 = 30}$

Apply $u(l) = 40$ in ① we get

$$u(l) = c_1(l) + c_2 = 40.$$

$$\Rightarrow c_1(60) + 30 = 40$$

$$c_1(60) = 40 - 30$$

$$60c_1 = 10$$

$$\boxed{c_1 = \frac{10}{60}}$$

$$\therefore ① \Rightarrow u(x) = \frac{1}{6}x + 30. \quad , \quad l = 60\text{ cm}$$

③ When the ends of a rod length 20 cm are maintained at the temperature 10°c and 20°c respectively until steady state is prevailed. Determine the steady state temperature of the rod.

Soln:

$$\frac{du}{dt} = 0$$

$$\Rightarrow \frac{d^2u}{dx^2} = 0$$

$$\Rightarrow u(x) = c_1x + c_2 \rightarrow ①$$

$$u(0) = 10 \ \& \ u(l) = 20°$$

$$\text{ie) } u(20) = 20°$$

$u(0) = 0$

$$① \Rightarrow c_1(0) + c_2 = 10$$

$$\boxed{c_2 = 10}$$

$$u(20) = 20$$

$$c_1(20) + 10 = 20$$

$$20c_1 = 10$$

$$\boxed{c_1 = 1/2}$$

$$\therefore \boxed{u(x) = \frac{1}{2}x + 10.}$$

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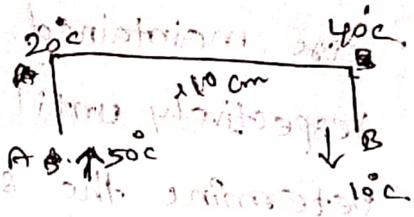
Steady state conditions and non-zero Boundary Conditions:-

① ~~A rod of length l cm~~

A bar 10 cm long with insulated sides, has its ends A and B kept at 20°c and 40°c respectively, until steady state conditions prevail. The temp at A is then suddenly raised to 50°c and at the same instant that at B is lowered to 10°c and maintained there after. Find the subsequent temp distribution in the bar.

Soln: one dimensional heat eqn is ~~The eqn to be solved is~~



$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \rightarrow Ⓐ$$

when the steady state condition prevails $\frac{\partial u}{\partial t} = 0$ and hence we get $\frac{\partial^2 u}{\partial x^2} = 0$

$\therefore$ Ⓐ reduces to $\frac{d^2 u}{dx^2} = 0$

Integrating we get $u(x) = px + q$

When $x = 0$	When $x = 10$
$u(0) = q$	$u(10) = p(10) + q$
$20° = q$	$40° = p(10) + 20°$
$\boxed{q = 20°}$	$10p = 20°$
	$\boxed{p = 2°}$

$\therefore$ $u(x) = 2x + 20$.

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The boundary conditions are

(i) $u(0,t) = 20 \quad \forall\ t \geq 0$

(ii) $u(10,t) = 40 \quad \forall\ t \geq 0$

(iii) $u(x,0) = f(x) = 2x + 20 \quad \forall\ x$.

The suitable soln is

$$u(x,t) = (A\cos px + B\sin px)\, e^{-\alpha^2 p^2 t} \rightarrow (1)$$

Applying condition (i) in (1), we get

$$u(0,t) = \cancel{(A\cos 0 p} \, A\, e^{-\alpha^2 p^2 t} = 20 \rightarrow (2)$$

Applying condition (ii) in (1) we get

$$u(10,t) = (A\cos 10p + B\sin 10p)\, e^{-\alpha^2 p^2 t} = 40 \rightarrow (3)$$

From (2) & (3) is not possible to find the constants A and B

since we've infinite number of values for A and B.

∴ in this case we split the soln $u(x,t)$ into two parts

$$u(x,t) = u_s(x) + u_T(x,t) \longrightarrow (4)$$

where $u_s(x)$ is a soln of the eqn.

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$ and is a function of x alone

Satisfying the conditions.

$$u_s(0) = c \quad \text{and} \quad u_s(l) = d$$

$u_T(x,t)$ is a transient soln satisfying (4)

which decreases as t increases.

If $u(x,t)$ is the subsequence temperature function the boundary conditions are

i) $u(0,t) = 50.$

(ii) $u(10,t) = 10.$

(iii) $u(x,0) = \frac{40-20}{10}x + 20$

$= 2x + 20$

To find $u_s(x)$:-

$u_s(x) = Ax + B$

put $x=0$ we get

$u_s(0) = B$

$\boxed{B = 50}$

put $x=10$ we get

$u_s(10) = A(10) + B$

$10 = 10A + 50$

$\boxed{A = -4}$

$\therefore u_s(x) = -4x + 50.$

To find $u_T(x,t)$:-

(4) $\Rightarrow u(x,t) = u_s(x) + u_T(x,t)$

$u_T(x,t) = u(x,t) - u_s(x) \rightarrow (5)$

put $x=0$ in eqn (5) we get

$u_T(0,t) = u(0,t) - u_s(0)$

$\therefore u_T(0,t) = 50 - 50$

$= 0.$

put $x=10$ in (5) we get

$u_T(10,t) = u(10,t) - u_s(10)$

$\therefore u_T(10,t) = 10 - 10$

$= 0.$

Put $t=0$ in ⑤ we get

$$u_T(x,0) = u(x,0) - u_s(x)$$
$$= (2x+20) - (-4x+50)$$
$$= 6x - 30$$

∴ given new boundary conditions are

(i) $u_T(0,t) = 0 \quad \forall \; t>0$

(ii) $u_T(10,t) = 0 \quad \forall \; t>0$

(iii) $u_T(x,0) = 6x - 30$

Now, the suitable soln is

$$u_T(x,t) = (A\cos px + B\sin px)\, e^{-\alpha^2 p^2 t} \rightarrow Ⓐ$$

Applying condition (i) in Ⓐ we get

$$u_T(0,t) = A e^{-\alpha^2 p^2 t} = 0$$

Here $e^{-\alpha^2 p^2 t} \neq 0$ [∵ It's defined $\forall t$]

$$\therefore A = 0$$

Substitute $A=0$ in Ⓐ we get

$$u_T(x,t) = B\sin px \; e^{-\alpha^2 p^2 t} \rightarrow ⑤$$

Applying condition (ii) in Ⓑ we get

$$u_T(10,t) = B\sin 10p\, e^{-\alpha^2 p^2 t} = 0.$$

Here $e^{-\alpha^2 p^2 t} \neq 0$ [∵ It is defined $\forall t$]

$B \neq 0$

[If $B=0$ & $A=0$ then we get trivial soln]

$\therefore \sin 10p = 0$

$\sin 10p = \sin n\pi$

$10p = n\pi$

$$\boxed{p = \frac{n\pi}{10}}$$

Substitute $p = \frac{n\pi}{10}$ in (B) we get

$$u_T(x,t) = B \sin\frac{n\pi x}{10}\, e^{-\frac{\alpha^2 n^2 \pi^2}{100} t} \rightarrow (C)$$

The most general soln is

$$u_T(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10}\, e^{-\frac{\alpha^2 n^2 \pi^2}{100} t} \longrightarrow (D)$$

Now Applying condition (iii) in (D) we get

$$u_T(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} = 6x - 30 \longrightarrow (E)$$

To find B_n, we expand $f(x)$ in a half range fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \longrightarrow (F)$$

where $b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$

From (E) & (F) we get $B_n = b_n$.

$$B_n = \frac{2}{10}\int_0^{10} (6x - 30) \sin\left(\frac{n\pi x}{10}\right) dx$$

$$= \frac{6}{5}\int_0^{10} (x-5) \sin\left(\frac{n\pi x}{10}\right) dx$$

$$= \frac{6}{5}\left[(x-5)\left(\frac{-\cos\frac{n\pi x}{10}}{n\pi/10}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^2}\right)\right]_0^{10}$$

$$= \frac{6}{5}\left[-(x-5)\frac{10}{n\pi}\cos\frac{n\pi x}{10} + \sin\frac{n\pi x}{10}\left(\frac{10}{n\pi}\right)^2\right]_0^{10}$$

$$= \frac{6}{5}\left[\left((-5)\frac{10}{n\pi}(-1)^n + 0\right) - \left(5\left(\frac{10}{n\pi}\right) + 0\right)\right]$$

$$= \frac{6}{5}\left[\frac{50}{n\pi}(-1)^{n+1} - \frac{50}{n\pi}\right] = \frac{6}{5}\left(\frac{50}{n\pi}\right)\left((-1)^{n+1} - 1\right)$$

$$B_n = \frac{60}{n\pi}\left[(-1)^{n+1} - 1\right]$$

Sub: the value of B_n in (D) we get

$$u_T(x,t) = \sum_{n=1}^{\infty} \frac{60}{n\pi}\left[(-1)^{n+1} - 1\right] \sin\frac{n\pi x}{10}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}}$$

$$= \frac{60}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1} - 1}{n}\right] \sin\frac{n\pi x}{10}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}}$$

Hence $u(x,t) = u_s(x) + u_T(x,t)$

$$= -4x + 50 + \frac{60}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} - 1}{n} \sin\frac{n\pi x}{10}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}}$$

$$u(x,t) = -4x + 50 - \frac{60}{\pi} \sum_{n=1}^{\infty} \left[\frac{1 + (-1)^n}{n}\right] \sin\frac{n\pi x}{10}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}}$$

A

Z- Transform

1. $Z[k]$ 2. $Z[a^n]$ 3. $Z[n]$ 4. $Z[na^n]$ 5. $Z[n^2]$

6. $Z\left[\frac{1}{n!}\right]$ 7. $Z\left[\frac{1}{n}\right]$ 8. $Z\left[\frac{1}{n+1}\right]$ 9. $Z\left[\frac{1}{n+2}\right]$

10. $Z[n+2]$ 11. $Z[n(n-1)]$ 12. $Z\left[\cancel{\cos n\theta}\ \frac{1}{n(n+1)}\right]$

13. $Z\left[\frac{1}{(n+1)(n+2)}\right]$ 14. $Z\left[\frac{2n+3}{(n+1)(n+2)}\right]$ 15. $Z[\cos n\theta]$ & $Z[\sin$

16. $Z\left[\cos \frac{n\pi}{2}\right]$ 17. $Z\left[\sin\frac{n\pi}{2}\right]$ 18. $Z[a^n \cos n\theta]$.

19. Initial and Final value Theorem.

20. Defn Convolution Thm. (i) $Z[2^n * n]$ (ii) $Z\{\cos\frac{n\pi}{2} * \sin$

21. $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$

22. $Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]$

23. P.T $Z^{-1}\left[\left(\frac{z}{z-a}\right)^2\right] = (n+1)a^n$

24. $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z-1)}\right]$

(or)

$Z^{-1}\left[\frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{4})}\right]$

25. Partial fraction :-

(i) Find $Z^{-1}\left[\frac{z}{z^2+7z+10}\right]$ (ii) $Z^{-1}\left[\frac{z^2+z}{(z-1)(z^2+1)}\right]$

Difference eqn

26. (i) $y_n = a + b3^n$ (ii) $y_n = a2^n + b(-2)^n$

Solution of Difference eqns

27. $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$ with $y_0 = 0$, $y_1 = 1$.

28. $y_{n+2} - 4y_n = n-1$, $y_0 = 0$ and $y_1 = 0$.

29. $y(k+2) - 4y(k+1) + 4y(k) = 0$ with $y(0) = 1$, $y(1) = 0$.

UNIT - V

	$x(n)$	$Z\{x(n)\} = X(z)$
1.	1	$\frac{z}{z-1}$, $\lvert z\rvert > 1$
2.	a^n	$\frac{z}{z-a}$, $\lvert z\rvert > \lvert a\rvert$
3.	$(-a)^{n}$	$\frac{z}{z+a}$
4.	a^{n-1}	$\frac{1}{z-a}$
5.	n	$\frac{z}{(z-1)^2}$
6.	n^2	$\frac{z^2+z}{(z-1)^3}$
7.	n^3	$\frac{z(z^2+4z+1)}{(z-1)^4}$
8.	n^k	$-z\frac{d}{dz} z(n^{k-1})$
9.	$n(n-1)$	$\frac{2z}{(z-1)^3}$
10.	$n(n-1)(n-2)$	$\frac{6z}{(z-1)^4}$
11.	$\frac{1}{n}$	$\log\left(\frac{z}{z-1}\right)$, $\lvert z\rvert > 1$, $n > 0$.
12.	$\frac{1}{n+1}$	$z\log\left(\frac{z}{z-1}\right)$
13.	$\frac{1}{n-1}$	$\frac{1}{z}\log\left(\frac{z}{z-1}\right)$, $n > 1$

	$x(n)$	$Z\{x(n)\} = X(z)$
	$\frac{1}{n!}$	$e^{1/z}$
15.	$\frac{1}{(n+1)!}$	$ze^{1/z} - z$
16.	k	$k\left[\frac{z}{z-1}\right]$
17.	$(-1)^n$	$\frac{z}{z+1}$
18.	$\left(\frac{1}{a}\right)^n$	$\frac{az}{az-1}$
19.	e^{an}	$\frac{z}{z-e^{a}}$
20.	e^{-an}	$\frac{z}{z-e^{-a}}$
21.	$\cos n\theta$	$\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$, $\lvert z\rvert>1$

28.	na^n	$\frac{az}{(z-a)^2}$
	n^2a^n	$\frac{za(z+a)}{(z-a)^3}$
29.	$a^n \frac{1}{n!}$	$e^{a/z}$
30.	$a^n \frac{1}{n}$	$\log \frac{z}{z-a}$
31.	$a^n r^n \cos n\theta$	$\frac{z(z-ar\cos\theta)}{z^2-2arz\cos\theta+a^2r^2}$
32.	$a^{n-1}(n-1)$	$\frac{z(2a-z)}{a(z-a)^2}$
33.	$a^n f(n)$	$F\left(\frac{z}{a}\right)$
34.	$n f(n)$	$-z\frac{d}{dz}X(z)$.

UNIT- V

Z- Transform & Difference Eqns

Z- Transform

Introduction:

Z- Transform plays an important role in communication engineering, digital signal processing and control systems. It is used in the analysis and representation of discrete time linear shift invariant systems.

Difference between Z- Transform & Laplace Transform

$$(1+x)^{-1} = 1-x+x^2-x^3+\cdots \qquad |x|<1$$

$$(1-x)^{-1} = 1+x+x^2+x^3+\cdots \qquad |x|<1$$

$$(1+x)^{-2} = 1-2x+3x^2-4x^3+\cdots \qquad |x|<1$$

$$(1-x)^{-2} = 1+2x+3x^2+4x^3+\cdots \qquad |x|<1$$

$$\log(1+x) = x-x^2/2+x^3/3-\frac{x^4}{4}+\cdots \qquad (-1\le x\le 1)$$

$$\log(1-x) = -\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\infty\right] \qquad (-1\le x\le 1).$$

$$\log\left(\frac{1+x}{1-x}\right) = 2\left[x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right]$$

$$\frac{e+e^{-1}}{2} = 1+\frac{1}{2!}+\frac{1}{4!}+\cdots$$

$$\frac{e-e^{-1}}{2} = 1+\frac{1}{3!}+\frac{1}{5!}+\cdots$$

Z-Transforms and Difference equations

Z-Transform, elementary Properties of z-transform.

Z-Transform: (two sided (or) bilateral).

Let $\{x(n)\}$ be a sequence defined for all integers then its z-transform is defined to be

$$z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

Z-Transform:-

Let $\{x(n)\}$ be a sequence defined for $n = 0, 1, 2, \ldots$ and $x(n) = 0$ for $n < 0$, then its z-transform is defined to be

$$z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n} = X(z).$$

where z is an arbitrary complex number

Z-Transform for discrete values of t.

If $f(t)$ is a function defined for discrete values of t where $t = nT$, $n = 0, 1, 2, 3, \ldots$ T being the sampling period, then z-transform of $f(t)$ is defined as

$$z\{f(t)\} = F(z) = \sum_{n=0}^{\infty} f(nT) z^{-n}.$$

Note:

1. mostly we study one-sided z- Transform
2. If f(n) given then replace 'n' by 'n'.
3. If f(t) given then replace 't' by 'nT'.
4. The double braces { } are used for a sequence

Sometimes we use [ ] or ( ).

Problems:

1. Find the Z-Transform of 1.

W.K.T $z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$.

$$z[1] = \sum_{n=0}^{\infty} 1 . z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n} = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots$$

$$1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \left[\frac{a}{z}\right]^n$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \cdots$$

$$= \left[1 - \frac{a}{z}\right]^{-1}$$

$$= \left[\frac{z-a}{z}\right]^{-1}$$

$$z\left[a^n\right] = \frac{z}{z-a}, \quad |z| > |a|$$

Note: $z\left[(-a)^n\right] = \frac{z}{z+a}$.

z-Transform of standard sequences:-

1. z-Transform of Right sided exponential sequence:

$$x(n) = a^n u(n) \text{ (or) } x(n) = \begin{cases} a^n & \text{for } n \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$z\left[a^n u(n)\right] = \frac{z}{z-a} \text{ (or) } \frac{1}{1-az^{-1}}, \quad \text{ROC}: |z| > |a|.$$

2. z-Transform of unit step sequence:-

$$x(n) = u(n) \quad \text{(or) } x(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$z\left[x(n)\right] = z\left[u(n)\right] = \frac{z}{z-1} \text{ (or) } \frac{1}{1-z^{-1}}$$

ROC: $|z| > 1$.

3. z-Transform of unit sample sequence $\delta(n)$:-

$$x(n) = \delta(n) = \begin{cases} 1, & \text{for } n=0 \\ 0, & \text{elsewhere} \end{cases}$$

$$Z[\delta(n)] = 1, \quad \text{ROC: Entire z-plane}$$

Properties of Z-Transform :-

1. Linearity :-

If $Z[x_1(n)] = X_1(z)$ and

$Z[x_2(n)] = X_2(z)$ then

$$Z[a_1 x_1(n) + a_2 x_2(n)] = a_1 X_1(z) + a_2 X_2(z)$$

2. Time Shifting :-

If $Z[x(n)] = X(z)$ then $Z[x(n-k)] = z^{-k} X(z)$

5. Differentiation in Z-Domain :-

If $Z[x(n)] = X(z)$ then

$$Z[nx(n)] = -z \frac{d}{dz} Z[-x(n)]$$

$$Z[nx(n)] = -z \frac{d}{dz} X(z).$$

6. Convolution in Time domain :-

If $Z[x_1(n)] = X_1(z)$ and $Z[x_2(n)] = X_2(z)$

then $Z[x_1(n) * x_2(n)] = X_1(z) X_2(z)$.

where $x_1(n) * x_2(n) = \sum_{k=0}^{n} x_1(k)\, x_2(n-k)$

7. Second shifting property (or) Second shifting theorem:

If $Z[x(n)] = X(z)$, then

$$Z[x(n+k)] = z^k \left[X(z) - x(0) - \frac{x(1)}{z} - \frac{x(2)}{z^2} - \cdots - \frac{x(k-1)}{z^{k-1}}\right]$$

Note:

1. z transform of delayed unit sample sequences

$Z[\delta(n-k)] = z^{-k}$ and

$Z[\delta(n+k)] = z^{k}$

2. z- transform of unit ramp sequence

$$Z[nu(n)] = \frac{z^{-1}}{(1-z^{-1})^2}$$

Initial value Theorem (IVT)

If $Z[f(n)] = F(z)$ then $f(0) = \lim_{z\to\infty} F(z)$

Final Value Theorem (FVT)

If $Z[f(n)] = F(z)$, then $\lim_{n\to\infty} f(n) = \lim_{z\to 1} (z-1)F(z)$

① Find the Z-Transform of the following sequences $\{x(n)\}$ whose $x(n)$ is given by

(i) $x(n) = k$ (ii) $x(n) = (-1)^n$ (iii) $x(n) = n$

(iv) $x(n) = na^n$ (v) $x(n) = n^2$ (vi) $x(n) = n(n-1)$ (or) $n^{(2)} - n$

(vii) $x(n) = \cos n\theta$ (viii) $x(n) = \sin n\theta$

(ix) $x(n) = r^n \sin n\theta$ x) $x(n) = r^n \cos n\theta$

Soln:

1. $Z[K] = \sum_{n=0}^{\infty} K z^{-n} = K \sum_{n=0}^{\infty} \frac{1}{z^n}$

$= K\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right]$

$= K\left[1 - \frac{1}{z}\right]^{-1}$

$= K\left[\frac{z-1}{z}\right]^{-1}$

$Z[K] = \frac{Kz}{z-1}$ if $|z| > 1$

(or) Soln:

W.K.T $Z[1] = \frac{z}{z-1}$

$Z[K] = K\, Z[1]$

$Z[K] = K\left[\frac{z}{z-1}\right]$

$Z[K] = \frac{Kz}{z-1}$

② $Z[(-1)^n] = \sum_{n=0}^{\infty} (-1)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{-1}{z}\right)^n$

$= 1 - \frac{1}{z} + \frac{1}{z^2} - \cdots$

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(or)

Soln:

W.K.T $z[a^n] = \frac{z}{z-a}$

Here $a=-1$

$z[(-1)^n] = \frac{z}{z-(-1)}$

$= \frac{z}{z+1}$

$= \left[1+\frac{1}{z}\right]^{-1}$

$= \left[\frac{z+1}{z}\right]^{-1}$

$= \frac{z}{z+1}$

2 (a) $z\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$

$= \frac{1}{1} + \frac{1}{1!}\frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \cdots$

$= e^{1/z}$

③ $z[n] = \sum_{n=0}^{\infty} n z^{-n} = \sum_{n=0}^{\infty} \frac{n}{z^n}$

$= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$

$= \frac{1}{z}\left[1+\frac{2}{z}+\frac{3}{z^2}+\cdots\right]$

$= \frac{1}{z}\left[1-\frac{1}{z}\right]^{-2}$

$= \frac{1}{z}\left[\frac{z-1}{z}\right]^{-2}$

$= \frac{1}{z} \cdot \frac{z^2}{(z-1)^2}$

$= \frac{z}{(z-1)^2}, \quad |z|>1.$

$z(n) = (-z)\frac{d}{dz} z[1]$

$= (-z)\frac{d}{dz}\left[\frac{z}{z-1}\right]$

$= (-z)\left[\frac{(z-1)\cdot 1 - z(1)}{(z-1)^2}\right]$

$= (-z)\left[\frac{z-1-z}{(z-1)^2}\right]$

$= \frac{z}{(z-1)^2}$

④ $z[na^n] = z[n]_{z \to \frac{z}{a}} = \left[\frac{z}{(z-1)^2}\right]_{z \to \frac{z}{a}} = \frac{az}{(z-a)^2}$

P.T $z\left[\frac{1}{n}\right] = \log\left(\frac{z}{z-1}\right)$ if $|z|>1$, $n>0$.

(5) $z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n} = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{z^n}$

$= \frac{1}{z} + \frac{1}{2}\frac{1}{z^2} + \frac{1}{3}\frac{1}{z^3} + \cdots$

$= \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \cdots$

$= -\log\left(1 - \frac{1}{z}\right)$

$= -\log\left(\frac{z-1}{z}\right) = \log\left(\frac{z-1}{z}\right)^{-1}$

$z\left[\frac{1}{n}\right] = \log\left(\frac{z}{z-1}\right)$

(6) P.T $z\left[\frac{1}{n+1}\right] = z\log\frac{z}{z-1}$

Soln:

$z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot \frac{1}{z^n}$

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(7) $Z\left[\frac{1}{(n+1)!}\right] = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^{-n}$

$= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\frac{1}{z}\right)^n$

$= \frac{1}{\lfloor 1} \left(\frac{1}{z}\right)^0 + \frac{1}{\lfloor 2} \left(\frac{1}{z}\right)^1 + \frac{1}{\lfloor 3} \left(\frac{1}{z}\right)^2 + \cdots$

$= \frac{1}{1!} + \frac{\left(\frac{1}{z}\right)}{2!} + \frac{(1/z)^2}{3!} + \cdots$

$= z\left[\frac{\left(\frac{1}{z}\right)}{1!} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{(1/z)^3}{3!} + \cdots\right]$

$= z\left[e^{\frac{1}{z}} - 1\right] = z e^{\frac{1}{z}} - z.$

(8) Find $z[n+2]$.

Soln:

$z[n+2] = z[n] + z[2]$

$= z[n] + 2z[1]$

$= \frac{z}{(z-1)^2} + 2 \cdot \frac{z}{z-1} = \frac{z+2z(z-1)}{(z-1)^2} = \frac{z+2z^2-2z}{(z-1)^2}$

$= \frac{2z^2-z}{(z-1)^2}$

(9) $z[n^2] = z[n \cdot n] = (-z)\frac{d}{dz} z[n]$

$= (-z)\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$

$= (-z)\left[\frac{(z-1)^2 \cdot 1 - z\,2(z-1)}{(z-1)^4}\right]$

$= (-z)\left[\frac{z-1-2z}{(z-1)^3}\right]$

$$= (-z)\left[\frac{-z-1}{(z-1)^3}\right]$$

$$= \frac{z(z+1)}{(z-1)^3}$$

(10) $z[n(n-1)] = z[n^2-n] = z[n^2] - z[n]$

$$= \frac{z^2+z}{(z-1)^3} - \frac{z}{(z-1)^2}$$

$$= \frac{z^2+z-z(z-1)}{(z-1)^3}$$

$$= \frac{z^2+z-z^2+z}{(z-1)^3} = \frac{2z}{(z-1)^3}$$

(11) Find $z[e^{an}]$

Soln:

$$z[e^{an}] = z[(e^a)^n] = \frac{z}{z-e^a}$$ Here $a = e^a$.

(12) Find $z[\cos n\theta]$ and $z[\sin n\theta]$

Soln:

Let $a = e^{i\theta}$

$$a^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta$$

W.K.T $z[a^n] = \frac{z}{z-a}$

$$z[(e^{i\theta})^n] = \frac{z}{z-e^{i\theta}}$$ Here $a = e^{i\theta}$

$$= \frac{z}{z-(\cos\theta+i\sin\theta)}$$

$$z[e^{in\theta}] = \frac{z}{(z-\cos\theta)-i\sin\theta}$$

$$z[e^{in\theta}] = \frac{z}{(z-\cos\theta)-i\sin\theta} \times \frac{(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)+i\sin\theta}$$

$$z[\cos n\theta + i\sin n\theta] = \frac{z[(z-\cos\theta)+iz\sin\theta}{(z-\cos\theta)^2+\sin^2\theta}$$

$$z[\cos n\theta]+iz[\sin n\theta] = \frac{z(z-\cos\theta)}{(z-\cos\theta)^2+\sin^2\theta} + i\frac{z\sin\theta}{(z-\cos\theta)^2+\sin^2\theta}$$

Equating the real and imaginary parts we get

$$z[\cos n\theta] = \frac{z(z-\cos\theta)}{(z-\cos\theta)^2+\sin^2\theta} \quad \& \quad z[\sin n\theta] = \frac{z\sin\theta}{(z-\cos\theta)^2+\sin^2\theta}$$

$$z[\cos n\theta] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+\cos^2\theta+\sin^2\theta}$$

~~$z[\sin n\theta]$~~

$$z[\sin n\theta] = \frac{z\sin\theta}{z^2-2z\cos\theta+\cos^2\theta+\sin^2\theta}$$

$$z[\cos n\theta] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}, \quad |z|>1$$

$$z[\sin n\theta] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}, \quad |z|>1.$$

(20) $z[na^n] = (-z)\frac{d}{dz} z[a^n]$

$= (-z)\frac{d}{dz}\left[\frac{z}{z-a}\right]$

$= (-z)\left[\frac{(z-a)-z}{(z-a)^2}\right]$

$= \frac{az}{(z-a)^2}$

(21) $z[n^3] = z[n \cdot n^2] = (-z)\frac{d}{dz} z[n^2]$

$= (-z)\frac{d}{dz}\left[\frac{z^2+z}{(z-1)^3}\right]$

$= (-z)\left[\frac{(z-1)^3(2z+1) - (z^2+z)3(z-1)^2}{(z-1)^6}\right]$

$= (-z)\left[\frac{(z-1)(2z+1) - 3(z^2+z)}{(z-1)^4}\right]$

$= (-z)\left[\frac{2z^2+z-2z-1-3z^2-3z}{(z-1)^4}\right]$

$= \frac{(-z)\left[-z^2-4z-1\right]}{(z-1)^4}$

$= \frac{z\left[z^2+4z+1\right]}{(z-1)^4}.$

(22) $z[n(n-1)(n-2)] = z[n^3 - 3n^2 + 2n]$

$= z[n^3] - 3z[n^2] + 2z[n]$

$= \frac{z[z^2+4z+1]}{(z-1)^4} - 3\frac{z(z+1)}{(z-1)^3} + 2\frac{z}{(z-1)^2}$

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$$= 1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \cdots$$

$$= z\left[\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$= z\left[-\log\left(1-\frac{1}{z}\right)\right]$$

$$= z\left[-\log\left(\frac{z-1}{z}\right)\right]$$

$$Z\left[\frac{1}{n+1}\right] = z\log\left(\frac{z}{z-1}\right)$$

(24) $$Z\left[\frac{1}{n+2}\right] = \sum_{n=0}^{\infty} \frac{1}{n+2}\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+2}\left(\frac{1}{z}\right)^{n}$$

$= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{4}\left(\frac{1}{z}\right)^2 + \cdots$

$= z^2\left[\frac{1}{2}\cdot\frac{1}{z^2} + \frac{1}{2}\cdot\frac{1}{z^3} + \frac{1}{4}\cdot\frac{1}{z^4} + \cdots\right]$

$= z^2\left[-\log\left(1-\frac{1}{z}\right) - \frac{1}{z}\right]$

$= z^2\left[-\log\left(\frac{z-1}{z}\right) - \frac{1}{z}\right]$

$= \cancel{z^2\left[-\log\right.} = z^2 \log\frac{z}{z-1} - z.$ 6, 31, 34, 60

(25) $z\left[\frac{a^n}{n!}\right] = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n}$

$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{a}{z}\right)^n$

$= 1 + \frac{1}{1!}\left(\frac{a}{z}\right) + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \cdots$

$= e^{a/z}$

(26) $z\left[(n+1)\,a^n\right] = z\left[na^n + a^n\right]$

$= z\left[na^n\right] + z\left[a^n\right]$

$= \frac{az}{(z-a)^2} + \frac{z}{z-a}$

$= \frac{az + z(z-a)}{(z-a)^2}$

$= \frac{\cancel{az} + z^2 - \cancel{az}}{(z-a)^2} = \frac{z^2}{(z-a)^2}$

If $Z[f(n)] = F(z)$ then $\lim_{n\to\infty}[f(n)] = \lim_{z\to 1}(z-1)F(z)$

proof:

By defn.

$$Z[f(n+1) - f(n)] = \sum_{n=0}^{\infty}[f(n+1) - f(n)]z^{-n}$$

$$Z[f(n+1)] - Z[f(n)] = \sum_{n=0}^{\infty}[f(n+1) - f(n)]z^{-n}$$

$$z[F(z) - f(0)] - F(z) = \sum_{n=0}^{\infty}[f(n+1) - f(n)]z^{-n}$$

$$(z-1)F(z) - zf(0) = \sum_{n=0}^{\infty}[f(n+1) - f(n)]z^{-n}.$$

Taking limit $z\to 1$, we get

$$\lim_{z\to 1}[(z-1)F(z)] - f(0) = \lim_{z\to 1}\sum_{n=0}^{\infty}[f(n+1) - f(n)]z^{-n}$$

$$\lim_{z\to 1}\left[(z-1)F(z)\right]-f(0)=\sum_{n=0}^{\infty}\left[f(n+1)-f(n)\right]$$

$$=\lim_{n\to\infty}\left[f(1)-f(0)+f(2)-f(1)+\cdots+f(n+1)-f(n)\right]$$

$$\lim_{z\to 1}(z-1)F(z)-\cancel{f(0)}=\lim_{n\to\infty}f(n+1)-\cancel{f(0)}$$

$$\lim_{z\to 1}(z-1)F(z)=\lim_{n\to\infty}f(n).$$

$\lim_{n\to\infty} f(n+1)$
$=\lim_{n\to\infty} f(n(1+\frac{1}{n}))$
$=\lim_{n\to\infty} f(n).$

① Find $Z[a^n \cos n\theta]$

Soln:

$$Z[a^n\cos n\theta]=Z[\cos n\theta]_{z\to\frac{z}{a}}$$

$$=\left[\frac{z^2-z\cos\theta}{z^2-2z\cos\theta+1}\right]_{z\to\frac{z}{a}}$$

$$=\frac{\left(\frac{z}{a}\right)^2-\frac{z}{a}\cos\theta}{\left(\frac{z}{a}\right)^2-2\frac{z}{a}\cos\theta+1}$$

$$=\frac{\frac{z}{a}-\cos\theta}{\frac{z}{a}-2\cos\theta+\frac{1}{(z/a)}}=\frac{(z-a\cos\theta)/\cancel{a}}{z^2-2az\cos\theta+a^2/\cancel{a}z}$$

$$=\frac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$$

$$=\frac{z-a\cos\theta}{z-2a\cos\theta+\frac{a}{z/a}}$$

$$=\frac{z-a\cos\theta}{\frac{z^2-2az\cos\theta+a^2}{z}}=\frac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$$

$$z[a^n \cos n\theta] = \frac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$$

III^{ly}

$$z[a^n \sin n\theta] = \frac{za\sin\theta}{z^2-2az\cos\theta+a^2}$$

② Find the z-transform of $x(n) = [3(4^n)-4(2^n)]u(n)$

Soln:

$$z[x(n)] = z[3(4^n)u(n)] - z[4(2^n)u(n)]$$

$$= 3z[4^n u(n)] - 4z[2^n u(n)]$$

W.K.T

$$z[a^n u(n)] = \frac{z}{z-a}$$

$u(n) = 1$ unit step function

$$z[x(n)] = 3.\frac{z}{z-4} - 4.\frac{z}{z-2}$$

$$= 3.\frac{1}{1-\frac{4}{z}} - 4.\frac{1}{1-\frac{2}{z}}$$

$$= \frac{3}{1-4z^{-1}} - \frac{4}{1-2z^{-1}}$$

ROC : $|z| > 4$

③ Find the z-transform of $f(n) = \frac{2n+3}{(n+1)(n+2)}$

Soln:

Given $f(n) = \frac{2n+3}{(n+1)(n+2)} = \frac{1}{n+1} + \frac{1}{n+2}$

$2n+3 = A(n+2)+B(n+1)$

$n=-2$

$-1 = -B \Rightarrow B=1$

$n=-1$

$1 = A$

$$z\left[\frac{1}{n+1}\right] = z\log\frac{z}{z-1}$$

$$z\left[\frac{1}{n+2}\right] = z^2\log\left(\frac{z}{z-1}\right) - z$$

$$\therefore z[f(n)] = z\log\left(\frac{z}{z-1}\right) + z^2\log\left(\frac{z}{z-1}\right) - z$$

$$= z(z+1)\log\left(\frac{z}{z-1}\right) - z.$$

Defn:

Convolution of sequences:-

The convolution of two sequences $\{x(n)\}$ and $\{y(n)\}$ is defined as $\{f(n) * g(n)\} = \sum_{m=0}^{n} f(m)\, g(n-m)$

$\{x(n) * y(n)\} = \sum_{m=0}^{n} x(m)\, y(n-m)$

Convolution theorem:-

i) If $z[x(n)] = X(z)$ and $z[y(n)] = Y(z)$

then $z[x(n) * y(n)] = X(z) \cdot Y(z)$

Note:

Inverse z- Transform using convolution Theorem:-

① Using Convolution theorem, find $z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$

Soln:

$$z[a^n] = \frac{z}{z-a} \Rightarrow z^{-1}\left[\frac{z}{z-a}\right] = a^n$$

$$z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-b}\right]$$

$$= z^{-1}\left[\frac{z}{z-a}\right] * z^{-1}\left[\frac{z}{z-b}\right]$$

$$= a^n * b^n$$

$$= \sum_{m=0}^{n} a^m b^{n-m}$$

$$= b^n \sum_{m=0}^{n} a^m b^{-m}$$

$$= b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m$$

$$= b^n\left[1 + \frac{a}{b} + \left(\frac{a}{b}\right)^2 + \cdots + \left(\frac{a}{b}\right)^n\right]$$

$$= b^n\left[\frac{(a/b)^{n+1} - 1}{a/b - 1}\right] = \frac{a^{n+1} - b^{n+1}}{a-b}.$$

Note: Geometric Progression

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r-1} \quad \text{if } r>1$$

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1-r} \quad \text{if } r<1$$

② Using Convolution theorem, evaluate $z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]$

Soln:

$$z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = z^{-1}\left[\frac{z}{z-1}\cdot\frac{z}{z-3}\right]$$

$$= z^{-1}\left[\frac{z}{z-1}\right] * z^{-1}\left[\frac{z}{z-3}\right]$$

$$= 1^n * 3^n \quad (\text{by Convolution thm})$$

$$= \sum_{m=0}^{n} 1^m 3^{n-m}$$

$$= 1 + 3 + \cdots + 3^{n-2} + 3^{n-1} + 3^n$$

$$= 1 + 3 + \cdots + 3^n \qquad \Big| \text{ Here } r = 3 > 1$$

$$= \frac{3^{n+1} - 1}{3 - 1}$$

$$z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = \frac{3^{n+1} - 1}{2}$$

(3) P.T $z^{-1}\left[\left(\frac{z}{z-a}\right)^2\right] = (n+1)a^n$

Soln:

$$z^{-1}\left[\left(\frac{z}{z-a}\right)^2\right] = z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-a}\right]$$

$$= z^{-1}\left[\frac{z}{z-a}\right] * z^{-1}\left[\frac{z}{z-a}\right]$$

$$= a^n * a^n$$

$$= \sum_{m=0}^{n} a^m a^{n-m}$$

$$= \sum_{m=0}^{n} a^n$$

$$= a^n + a^n + a^n + \cdots \ (n+1 \text{ terms})$$

$$z^{-1}\left[\left(\frac{z}{z-a}\right)^2\right] = (n+1)a^n.$$

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④ Find $z^{-1}\left[\frac{z^3}{(z-2)^2(z-3)}\right]$

Soln:

$$z^{-1}\left[\frac{z^3}{(z-2)^2(z-3)}\right] = z^{-1}\left[\frac{z^2}{(z-2)^2}\cdot\frac{z}{z-3}\right]$$

$$= z^{-1}\left[\frac{z^2}{(z-2)^2}\right] * z^{-1}\left[\frac{z}{z-3}\right]$$

$$= (n+1)2^n * 3^n$$

$$= \sum_{m=0}^{n}(m+1)2^m 3^{n-m}$$

$$= 3^n\sum_{m=0}^{n}(m+1)2^m 3^{-m}.$$

⑤ Find $z^{-1}\left[\frac{z^2}{(z+a)^2}\right]$

Soln:

$$z^{-1}\left[\frac{z^2}{(z+a)^2}\right] = z^{-1}\left[\frac{z}{z+a}\cdot\frac{z}{z+a}\right]$$

$$= z^{-1}\left[\frac{z}{z+a}\right] * z^{-1}\left[\frac{z}{z+a}\right]$$

$$= (-a)^n * (-a)^n$$

$$= \sum_{m=0}^{n}(-a)^m(-a)^{n-m}$$

$$= \sum_{m=0}^{n}(-a)^n$$

$$= (-a)^n + (-a)^n + \ldots \text{ ((n+1) terms)}$$

$$= (n+1)(-a)^n.$$

Inverse z-Transform
(Partial fraction method)

① Find the inverse z-transform of $\frac{z}{z^2-3z+2}$

Soln:-

let $X(z) = \frac{z}{z^2-3z+2}$

$$\frac{X(z)}{z} = \frac{1}{z^2-3z+2}$$

$z^2-3z+2=0$

$z = \frac{3 \pm \sqrt{9-4(1)2}}{2}$

$= \frac{3 \pm 1}{2}$

$= \frac{4}{2}, \frac{2}{2}$

$z = 2, 1$

$(z-2)(z-1)$

$$\frac{X(z)}{z} = \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

Put $z=1$ we get

$1 = -A$

$\boxed{A=-1}$

Put $z=2$ we get

$1 = B$

$\boxed{B=1}$

$$\frac{X(z)}{z} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$X(z) = \frac{-z}{z-1} + \frac{z}{z-2}$$

$$z[x(n)] = \frac{-z}{z-1} + \frac{z}{z-2}$$

$$x(n) = -z^{-1}\left[\frac{z}{z-1}\right] + z^{-1}\left[\frac{z}{z-2}\right]$$

$$= -(1)^n + (2)^n$$

$$z^{-1}\left[\frac{z}{z^2-3z+2}\right] = (2)^n - (1)^n.$$

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② Find the inverse z transform of $\frac{z}{z^2+7z+10}$

Soln:

Let $X(z) = \frac{z}{z^2+7z+10}$

$$\frac{X(z)}{z} = \frac{1}{z^2+7z+10} = \frac{1}{(z+2)(z+5)}$$

$$\frac{X(z)}{z} = \frac{1}{(z+2)(z+5)} = \frac{A}{z+2} + \frac{B}{z+5}$$

$z = \frac{-7 \pm \sqrt{49-40}}{2}$

$= \frac{-7 \pm 3}{2}$

$= \frac{-4}{2}, -5$

$z = -2, -5$

$(z+2)(z+5)$

$$1 = A(z+5) + B(z+2)$$

put $z=-2$ we get,

$1 = 3A$

$\boxed{A = 1/3}$

put $z=-5$ we get

$1 = -3B$

$\boxed{B = \frac{-1}{3}}$

$$\therefore \frac{X(z)}{z} = \frac{1}{3(z+2)} - \frac{1}{3(z+5)}$$

$$X(z) = \frac{1}{3} \cdot \frac{z}{z+2} - \frac{1}{3} \frac{z}{z+5}$$

$$z[x(n)] = \frac{1}{3} \frac{z}{z+2} - \frac{1}{3} \frac{z}{z+5}$$

$$x(n) = \frac{1}{3} z^{-1}\left[\frac{z}{z+2}\right] - \frac{1}{3} z^{-1}\left[\frac{z}{z+5}\right]$$

$$= \frac{1}{3}(-2)^n - \frac{1}{3}(-5)^n$$

$$z^{-1}\left[\frac{z}{z^2+7z+10}\right] = \frac{1}{3}\left[(-2)^n - (-5)^n\right]$$

(3) Find the inverse z-transform of $\frac{z^3}{(z-1)^2(z-2)}$

Soln:

Let $X(z) = \frac{z^3}{(z-1)^2(z-2)}$

$$\frac{X(z)}{z} = \frac{z^2}{(z-1)^2(z-2)} = \frac{A}{z-2} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$z^2 = A(z-1)^2 + B(z-1)(z-2) + C(z-2)$$

put $z=1$ we get

$$1 = A(0) + B(0) + C(1-2)$$

$$-C = 1$$

$$\boxed{C=-1}$$

Put $z=2$ we get

$$4 = A+0+0$$

$$\boxed{A=4}$$

put $z=0$ we get

$$0 = A + B(2) + C(-2)$$

$$4+2B-2(-1)=0$$

$$4+2B+2=0$$

$$2B+6=0$$

$$2B=-6$$

$$\boxed{B=-3}$$

Co eff of z^2
$A+B=1$
$B=-3$

$$\therefore \frac{X(z)}{z} = \frac{4}{z-2} - \frac{3}{z-1} - \frac{1}{(z-1)^2}$$

$$X(z) = 4 \cdot \frac{z}{z-2} - 3 \cdot \frac{z}{z-1} - \frac{z}{(z-1)^2}$$

$$z[x(n)] = 4\frac{z}{z-2} - 3\frac{z}{z-1} - \frac{z}{(z-1)^2}$$

$$x(n) = 4z^{-1}\left[\frac{z}{z-2}\right] - 3z^{-1}\left[\frac{z}{z-1}\right] - z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$x(n) = 4(2)^n - 3(1)^n - n.$$

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$$x(n) = 4(2)^n - 3(1)^n - n.$$

$$z^{-1}\left[\frac{z^3}{(z-1)^2(z-2)}\right] = 4(2)^n - 3(1)^n - n.$$

(4) Find the inverse z-transform of $\frac{z^2+z}{(z-1)(z^2+1)}$

Soln:

Let $X(z) = \frac{z^2+z}{(z-1)(z^2+1)}$

$$\frac{X(z)}{z} = \frac{z+1}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{Bz+C}{z^2+1}$$

$$z+1 = A(z^2+1) + (Bz+C)(z-1)$$

Put $z=1$ we get

$2 = 2A + (B+C)(0)$

$2A = 2$

$\boxed{A=1}$

Coefficient of z^2

$0 = A+B$

$\boxed{B=-1}$

Coefficient of z

$1 = 0 - B + C$

$C - B = 1$

$C + 1 = 1$

$\boxed{C=0}$

$z=0$

$1 = A - C$

$C = A - 1$

$C = A - 1 = 0$

$z=0$

$1 = A - C$

$C = A - 1$

$C = 1 - 1$

$C = 0$

(1) $\therefore \frac{X(z)}{z} = \frac{1}{z-1} - \frac{z}{z^2+1}$

$$X(z) = \frac{z}{z-1} - \frac{z^2}{z^2+1}$$

$$Z[x(n)] = \frac{z}{z-1} - \frac{z^2}{z^2+1}$$

$$x(n) = z^{-1}\left[\frac{z}{z-1}\right] - z^{-1}\left[\frac{z^2}{z^2+1}\right]$$

Hw

① $z^{-1}\left[\dfrac{z(z^2-z+2)}{(z+1)(z-1)^2}\right]$ Partial fraction method

② find $z^{-1}\ \dfrac{z^2}{(z+2)(z^2+4)}$ using partial fraction method.

Soln: let $X(z) = \dfrac{z^2}{(z+2)(z^2+4)}$

$$\frac{X(z)}{z} = \frac{z}{(z+2)(z^2+4)} = \frac{A}{z+2} + \frac{Bz+C}{z^2+4}$$

$$z = A(z^2+4) + (Bz+C)(z+2)$$

put $z=-2$	put $z=0$	coefficient of z^2
$-2 = 8A$	$0 = 4A+2C$	$0 = A+B$

$$\frac{X(z)}{z} = \frac{-1}{4}\frac{1}{z+2} + \frac{1}{4}\frac{z}{z^2+4} + \frac{1}{2}\frac{1}{z^2+4}$$

$$X(z) = \frac{-1}{4}\frac{z}{z+2} + \frac{1}{4}\frac{z^2}{z^2+4} + \frac{1}{4}\frac{2z}{z^2+4}$$

$$Z[x(n)] = \frac{-1}{4}\frac{z}{z+2} + \frac{1}{4}\frac{z^2}{z^2+4} + \frac{1}{4}\frac{2z}{z^2+4}$$

$$x(n) = \frac{-1}{4} Z^{-1}\left[\frac{z}{z+2}\right] + \frac{1}{4} Z^{-1}\left[\frac{z^2}{z^2+4}\right] + \frac{1}{4} Z^{-1}\left[\frac{2z}{z^2+4}\right]$$

$$x(n) = \frac{-1}{4}(-2)^n + \frac{1}{4} 2^n \cos\frac{n\pi}{2} + \frac{1}{4} 2^n \sin\frac{n\pi}{2}$$

$$\left[\begin{array}{l} Z\left[a^n \cos\frac{n\pi}{2}\right] = \frac{z^2}{z^2+a^2} \\ Z\left[a^n \sin\frac{n\pi}{2}\right] = \frac{az}{z^2+a^2} \end{array}\right.$$

2) Find $Z^{-1}\left[\frac{z(z^2-z+2)}{(z+1)(z-1)^2}\right]$

Let $X(z) = \frac{z(z^2-z+2)}{(z+1)(z-1)^2}$

$$\frac{X(z)}{z} = \frac{z^2-z+2}{(z+1)(z-1)^2} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$z^2-z+2 = A(z-1)^2 + (B)(z+1)(z-1) + C(z+1)$$

put $z=1$

$2 = 2C$

$\boxed{C=1}$

put $z=-1$,

$1+1+2 = 4A$

$4 = 4A$

$\boxed{A=1}$

Equating z^2 term

$1 = A+B$

$B = 1-A$

$B = 1-1$

$\boxed{B=0}$

$$\frac{X(z)}{z} = \frac{1}{z+1} + 0 + \frac{1}{(z-1)^2}$$

$$X(z) = \frac{z}{z+1} + \frac{z}{(z-1)^2}$$

$$z[x(n)] = \frac{z}{z+1} + \frac{z}{(z-1)^2}$$

$$x(n) = z^{-1}\left[\frac{z}{z+1}\right] + z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$x(n) = (-1)^n + n.$$

(3) Form the difference eqn from $y_n = a + b3^n$

Soln:

$$y_n = a + b3^n \longrightarrow (1)$$

$$y_{n+1} = a + b3^{n+1}.$$

$$y_{n+1} = a + 3b3^n \rightarrow (2)$$

$$y_{n+2} = a + 9b3^n \rightarrow (3)$$

From (1) & (2) & (3) we get

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 1 & 3 \\ y_{n+2} & 1 & 9 \end{vmatrix} = 0.$$

$$y_n(9-3) - y_{n+1}(9-1) + y_{n+2}(3-1) = 0$$

$$6y_n - 8y_{n+1} + 2y_{n+2} = 0$$

$$\therefore 2y_{n+2} - 8y_{n+1} + 6y_n = 0$$

$$y_{n+2} - 4y_{n+1} + 3y_n = 0.$$

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Formation of Difference equations:-

Defn:

Difference eqns:-

A differential eqn is a relation between the differences of an unknown function at one (or) more general values of the argument.

Thus $\Delta y_{(n+1)} + y_{(n)} = 2$

and $\Delta y_{(n+1)} + \Delta^2 y_{(n-1)} = 1$ } difference eqns.

(1) From $y_n = a2^n + b(-2)^n$, derive a difference eqn by eliminating the constants.

Soln: Given $y_n = a2^n + b(-2)^n$

$$y_{n+1} = a2^{n+1} + b(-2)^{n+1}$$

$$y_{n+1} = 2a2^n - 2b(-2)^n$$

$$y_{n+2} = a2^{n+2} + b(-2)^{n+2}$$

$$= a2^n 2^2 + b(-2)^n(-2)^2$$

$$y_{n+2} = 4a2^n + 4b(-2)^n$$

Eliminating a & b we get

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & -2 \\ y_{n+2} & 4 & 4 \end{vmatrix} = 0$$

$y_n(8+8) - y_{n+1}(4-4) + y_{n+2}(-2-2)$

$16y_n - 4y_{n+2} = 0$

$$y_n(8+8) - 1(4y_{n+1} + 2y_{n+2}) + 1(4y_{n+1} - 2y_{n+2}) = 0.$$

$$16y_n - \cancel{4y_{n+1}} - 2y_{n+2} + \cancel{4y_{n+1}} - 2y_{n+2} = 0.$$

$$\Rightarrow 16y_n - 4y_{n+2} = 0$$

$\boxed{y_{n+2} - 4y_n = 0}$ which is desired diff eqn.

(2) Derive the difference eqn from $y_n = (A+Bn)(-3)^n$.

Soln: Given $y_n = (A+Bn)(-3)^n \longrightarrow (1).$

$$y_n = A(-3)^n + Bn(-3)^n$$

$$y_{n+1} = A(-3)^{n+1} + B(n+1)(-3)^{n+1}$$

$$= A(-3)^1(-3)^n + B(n+1)(-3)^n(-3)^1$$

$y_n\left[-\cancel{27n}-54+\cancel{27n}+27\right]-1\left[9ny_{n+1}+18y_{n+1}+3ny_{n+2}+3y_{n+2}\right]+9ny_n+3ny_{n+2}=0$

$y_n[-27]-9ny_{n+1}-18y_{n+1}-3ny_{n+2}-3y_{n+2}+9ny_{n+1}+3ny_{n+2}=0$

$-3y_{n+2}-18y_{n+1}-27y_n=0$

$$\boxed{y_{n+2}+6y_{n+1}+9y_n=0}$$

~~Application of z-transform~~ Solution of difference equations:- Using z-Transform:-

$z[y_{n+1}]=zy(z)-zy(0)$

$z[y_{n+2}]=z^2y(z)-z^2y(0)-zy(1)$

$z[y_{n+3}]=z^3y(z)-z^3y(0)-z^2y(1)-zy(2)$

$z[y_{(n-k)}]=z^{-k}y(z)$. where $y(z)=z[y_n]$

① Solve $y_{n+2}+4y_{n+1}+3y_n=2^n$ with $y_0=0,\ y_1=1$ Using z-transform method.

Soln: Let $y(z)=z[y_n]$ be the z-transform of $[y_n]$

Then $z[y_{n+1}]=z[y(z)-y_0]=zy(z)$

$z[y_{n+2}]=z^2y(z)-z^2y(0)-zy_1$

$z[y_{n+2}]=z^2y(z)-z$

Given

$$y_{n+2} + 4y_{n+1} + 3y_n = 2^n \longrightarrow (1)$$

Taking z-transform on both sides of (1)

$$Z[y_{n+2}] + 4Z[y_{n+1}] + 3Z[y_n] = Z[2^n]$$

$$z^2Y(z) - z + 4[zY(z)] + 3Y(z) = \frac{z}{z-2}$$

$$z^2Y(z) - z + 4zY(z) + 3Y(z) = \frac{z}{z-2}$$

$$(z^2+4z+3)Y(z) = \frac{z}{z-2} + z$$

$$Y(z) = \frac{z + z(z-2)}{(z-2)(z^2+4z+3)} = \frac{z+z^2-2z}{(z-2)(z^2+4z+3)}$$

$$= \frac{z^2-z}{(z-2)(z^2+4z+3)}$$

Let $$\frac{Y(z)}{z} = \frac{z-1}{(z-2)(z^2+4z+3)}$$

$$\frac{Y(z)}{z} = \frac{z-1}{(z-2)(z+1)(z+3)} = \frac{A}{z-2} + \frac{B}{z+1} + \frac{C}{z+3}$$

$$z-1 = A(z+1)(z+3) + B(z-2)(z+3) + C(z-2)(z+1)$$

Put $z=-1$ we get

$$-2 = A(0)(-2) + B(-3)(2) + 0$$

$$-2 = 0 - 6B$$

$$6B = 2$$

$$\boxed{B = \frac{1}{3}}$$

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Put $z=-3$, we get

$-4 = 0+0+C(-5)(-2)$

$10C = -4$

$\boxed{C = -\frac{2}{5}}$

Put $z=0$ we get

$-1 = A(3)+B(-6)+C(-2)$

$3A-6B-2C=-1$

$3A-6(\frac{1}{3})-2(-\frac{2}{5})=-1$

$3A-2+\frac{4}{5}=-1$

$3A = -1+2-\frac{4}{5}$

$= 1-\frac{4}{5}$

$3A = \frac{5-4}{5} = \frac{1}{5}$

$\boxed{A = \frac{1}{15}}$

Put $z=2$

$2-1 = A(3)(5)$

$1 = 15A$

$\boxed{A = \frac{1}{15}}$

$\therefore \frac{Y(z)}{z} = \frac{1}{15}\cdot\frac{1}{z-2}+\frac{1}{3}\cdot\frac{1}{z+1}-\frac{2}{5}\cdot\frac{1}{z+3}$

$Y(z) = \frac{1}{15}\cdot\frac{z}{z-2}+\frac{1}{3}\frac{z}{z+1}-\frac{2}{5}\cdot\frac{z}{z+3}$

$Y(z) = z[y(n)]$

$y(n) = \frac{1}{15}z^{-1}\left[\frac{z}{z-2}\right]+\frac{1}{3}z^{-1}\left[\frac{z}{z+1}\right]-\frac{2}{5}z^{-1}\left[\frac{z}{z+3}\right]$

$y(n) = \frac{1}{15}(2)^n+\frac{1}{3}(-1)^n-\frac{2}{5}(-3)^n$

② Solve using z-transform $y_{n+2} - 4y_n = n-1$, $y_0 = 0$ and $y_1 = 0$.

Soln:

Given $y_{n+2} - 4y_n = n-1$

Taking z transform on bothsides

$$z[y_{n+2}] - 4z[y_n] = z[n-1]$$

$$z^2 y(z) - z^2 y(0) - z y(1) - 4y(z) = z[n] - z[1]$$

$$z^2 y(z) - z^2(0) - z(0) - 4y(z) = \frac{z}{(z-1)^2} - \frac{z}{z-1}$$

$$(z^2-4)\,y(z) = \frac{z - z(z-1)}{(z-1)^2}$$

$$= \frac{z - z^2 + z}{}$$

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Put $z=2$

$0 = 0 + B(4) + 0 + 0$

$4B = 0$

$\boxed{B=0}$

Put $z=-2$

$4 = A(-4)(9) + 0 + 0 + 0$

$-9A = 1$

$\boxed{A = \frac{-1}{9}}$

Put $z=1$

$1 = 0 + 0 + 0 + D(3)(-1)$

$-3D = 1$

$\boxed{D = -1/3}$

Put $z=0$

coefficient of z^2

$2 = A(-2)(1) + B(2)(1) + C(2)(-2)(-1) + D(2)(-2)$

$2 = -2A + 2B + 4C - 4D$

$1 = -A + B + 2C - 2D$

$-A + B + 2C - 2D = 1$

$\frac{1}{9} + 0 + 2C - 2(-1/3) = 1$

$2C = 1 - \frac{2}{3} - \frac{1}{9}$

$2C = \frac{9-6-1}{9}$

$2C = \frac{2}{9}$

$\boxed{C = \frac{1}{9}}$

$$\frac{Y(z)}{z} = \frac{-1}{9(z+2)} + \frac{0}{z-2} + \frac{1}{9(z-1)} - \frac{1}{3(z-1)^2}$$

$$Y(z) = \frac{-1}{9}\frac{z}{z+2} + \frac{1}{9}\frac{z}{z-1} - \frac{1}{3}\frac{z}{(z-1)^2}$$

$$z[y(n)] = \frac{-1}{9}\frac{z}{z+2} + \frac{1}{9}\frac{z}{z-1} - \frac{1}{3}\frac{z}{(z-1)^2}$$

$$y(n) = \frac{-1}{9} z^{-1}\left[\frac{z}{z+2}\right] + \frac{1}{9} z^{-1}\left[\frac{z}{z-1}\right] - \frac{1}{3} z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$y(n) = \frac{-1}{9}(-2)^n + \frac{1}{9}(1)^n - \frac{1}{3}n.$$

(1)

Inverse z- Transform by inverse integral method:-
[Cauchy's Residue Thm].

From the relation between the z- Transform and Fourier transform of a sequence we get

$$x(n) = \frac{1}{2\pi i}\int_C X(z)\, z^{n-1}\, dz.$$

By Cauchy's Residue Thm

$$\int_C X(z)\, z^{n-1}\, dz = 2\pi i \text{ (sum of the residues } X(z)z^{n-1} \text{ at the isolated singularities)}$$

$\therefore$ $x(n)$ = sum of the residues of $X(z)z^{n-1}$ at the isolated singularities.

Note: Take the contour C such that all the poles of the function z lie within the ~~contor~~ contour.

i) If $z=a$ is a pole of order 1, ie) simple pole.

$$\underset{z=a}{\text{Res}}\, f(z) = \lim_{z\to a}(z-a)\,f(z).$$

ii) If $z=a$ is a pole of order m

$$\underset{z=a}{\text{Res}}\, f(z) = \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z)$$

(1) Find $z^{-1}\left[\dfrac{10z}{(z-1)(z-2)}\right]$

Let $X(z) = \dfrac{10z}{(z-1)(z-2)}$

$$X(z)\,z^{n-1} = \frac{10z}{(z-1)(z-2)}\,z^{n-1}$$

$$= \frac{10z^n}{(z-1)(z-2)}$$

To find Residues:-

Let $10/z$ be a contour

$(z-1)(z-2) = 0$

$z = 1, 2$ is a simple poles

$$\underset{z=1}{\text{Res}}\, X(z)z^{n-1} = \lim_{z\to 1} (z-1)\frac{10z^n}{(z-1)(z-2)}$$

$$= \frac{10}{1-2} = -10$$

$$\underset{z=2}{\text{Res}}\, (X(z)z^{n-1}) = \lim_{z\to 2} (z-2)\frac{10z^n}{(z-1)(z-2)}$$

$$= \frac{10(2)^n}{2-1} = 10(2)^n$$

$x(n) =$ Sum of Residues of $X(z)z^{n-1}$

$= -10 + 10(2)^n$

$$\boxed{x(n) = 10(2^n - 1)}$$

HW ① Find $z^{-1}\left[\frac{z(z+1)}{(z-1)^3}\right]$

Ans: $x(n) = n^2$.

② State and Prove Convolution Thm

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} y(n-k)\, z^{-n}$$

By changing the order of summation

$$= \sum_{k=-\infty}^{\infty} x(k) \left[\sum_{m=-\infty}^{\infty} y(m)\, z^{-(m+k)} \right]$$ by putting $n-k=m$

$$= \sum_{k=-\infty}^{\infty} x(k)\, z^{-k} \left[\sum_{m=-\infty}^{\infty} y(m)\, z^{-m} \right]$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, z^{-k} \sum_{m=-\infty}^{\infty} y(m)\, z^{-m}$$

$Z[x(n) * y(n)] = X(z) \cdot Y(z)$