KNOWLEDGE INSTITUTE OF TECHNOLOGY, SALEM
MA6351 -TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS
QUESTION BANK
(COMMON TO ALL BRANCHES)

Unit I - Partial Differential Equations
PART A

1. Form the PDE by eliminating the constants a and b from $z = ax + by + \sqrt{a^2 + b^2}$

Soln: Given that $z = ax + by + \sqrt{a^2 + b^2}$ **……..(i)**

differentiate w.r.t x: $\frac{\partial z}{\partial x} = p = a + 0 + 0 \Rightarrow p = a$

differentiate w.r.t y : $\frac{\partial z}{\partial y} = q = b \Rightarrow q = b$.

Substitute a and b in **(i)**

$\therefore z = px + qy + \sqrt{p^2 + q^2}$ Which is required PDE.

2. Form the PDE by eliminating the arbitrary constant a and b from $z = (x^2 + a^2)(y^2 + b^2)$

Soln : Given that $z = (x^2 + a^2)(y^2 + b^2)$ **…………(i)**

differentiate w.r.t x: $\frac{\partial z}{\partial x} = p = (2x)(y^2 + b^2)$ **……..(ii)**

differentiate w.r.t y : $\frac{\partial z}{\partial y} = q = (2y)(x^2 + a^2)$ **…..(iii)**

substitute (ii) and (iii) in (i) $\therefore z = \frac{p}{2x} \cdot \frac{q}{2y} \Rightarrow 4xyz = pq$

which is required PDE.

3. Form the PDE by eliminating the arbitrary constant a and b from $(x-a)^2 + (y-b)^2 + z^2 = 1$**.**

Soln : Given that $(x-a)^2 + (y-b)^2 + z^2 = 1$ **……….(i)**

differentiate w.r.t x: $2(x-a) + 2zp = 0$ **……..(ii)**

differentiate w.r.t y: $2(y-b) + 2zq = 0$ **…(iii)**

from (i) , (ii) and (iii) $\Rightarrow z^2p^2 + z^2q^2 + z^2 = 1$

$\therefore z^2(p^2 + q^2 + 1) = 1$. Which is required PDE.

4. Form the PDE by eliminating the arbitrary function from $z = f(x^2 + y^2)$

Soln: Given that $z = f(x^2 + y^2)$ **……….(i)**

differentiate w.r.t x: $\frac{\partial z}{\partial x} = p = f'.2x$ **……..(ii)**

differentiate w.r.t y : $\frac{\partial z}{\partial y} = q = f'.2y$ **…..(iii)**

Eliminate f **:** $\frac{(ii)}{(iii)} \Rightarrow \frac{p}{q} = \frac{2x}{2y} \quad \therefore py - qx = 0$

5. Form the PDE by eliminating the arbitrary function from $z = f\left(\frac{y}{x}\right)$

Soln: Given that $z = f\left(\frac{y}{x}\right)$ **…. (i)**

differentiate w.r.t x: $\frac{\partial z}{\partial x} = p = f'\left(y/x\right).\left(-y/x^2\right)$ **…(ii)**

differentiate w.r.t y: $\frac{\partial z}{\partial y} = q = f'\left(y/x\right).\left(1/x\right)$ **…..(iii)**

Eliminate f **:** $\frac{(ii)}{(iii)} \Rightarrow \frac{p}{q} = \frac{-y}{x} \Rightarrow px + qy = 0$ **.**

6. Form the PDE by eliminating the arbitrary function from $z = f\left(\frac{xy}{z}\right)$**.**

Soln: Given that $z = f\left(\frac{xy}{z}\right)$ **..........(i)**

differentiate w.r.t x : $\frac{\partial z}{\partial x} = p = f'\left(\frac{xy}{z}\right).\left(\frac{zy - xyp}{z^2}\right)$**........(ii)**

differentiate w.r.t y : $\frac{\partial z}{\partial y} = q = f'\left(\frac{xy}{z}\right).\left(\frac{zx - xyq}{z^2}\right)$ **.....(iii)**

Eliminate f **:**

$\frac{(ii)}{(iii)} \Rightarrow \frac{p}{q} = \frac{y(z - xp)}{x(z - yq)} \Rightarrow px(z - yq) = q\,y(z - xp) \quad \therefore xp = yq$

7. Form the PDE of all spheres whose centre lie on Z- axis .

Soln: Equation of the spheres with centre on Z – axis is

$x^2 + y^2 + (z - c)^2 = r^2$ …… (i)

Centre (0 , 0 ,c) and radius = r .

differentiate w.r.t x : 2x + 2 (z-c) P =0 ………(ii)

differentiate w.r.t y : 2y + 2 (z-c) q =0 ……….(iii)

$\frac{(ii)}{(iii)} \Rightarrow \frac{p}{q} = \frac{x}{y} \quad \therefore py - qx = 0.$

8. Find the PDE of the all plane having equal intercepts on the x and y axis .

Soln: Equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ……..(i)

Since x intercept = y intercept $\Rightarrow a = b$

$\therefore$ *from* (*i*) $\frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1$…….(ii)

Differentiate (ii) w. r.to x : $\frac{1}{a} + \frac{1}{c}p = 0$ ….(iii)

Differentiate (ii) w. r.to y: $\frac{1}{a} + \frac{1}{c}q = 0$ ….(iv)

Compare (iii) and (iv) $p = q$. This is required PDE.

9. Find the complete integral of $\sqrt{p} + \sqrt{q} = 1$.

Soln: Given that $\sqrt{p} + \sqrt{q} = 1$

$\Rightarrow f(p,q) = \sqrt{p} + \sqrt{q} - 1 = 0$

The solution is $z = ax + by + c$. ……..(i)

Replace p = a & q = b

Put $f(a,b) = 0 \Rightarrow \sqrt{a} + \sqrt{b} - 1 = 0$ *ie.,* $\sqrt{b} = (1 - \sqrt{a})^2$ ….(ii)

Substitute (ii) in (i) $\therefore z = ax + (1 - \sqrt{a})^2 y + c$ .This is complete solution.

10. Find the Complete and singular integral of $pq + p + q = 0$.

Soln: Given $pq + p + q = 0 \Rightarrow f(p,q) = 0$ *ie.,* $pq + p + q = 0$

The solution is $z = ax + by + c$. …….. (i)

Put $f(a,b) = 0$ *ie.,* $ab + a + b = 0 \Rightarrow b = \frac{-a}{a+1}$ …… (ii)

Substitute (ii) in (i) $\therefore z = ax - \frac{a}{a+1}y + c$. This is complete solution. Differentiate w.r.to 'c' both sides 0 = 1 (absurd) . $\therefore$ No singular solution.

11. Obtain the Complete integral of $z = px + qy - 2\sqrt{pq}$

Soln: The Clairaut's form is $z = px + qy + f(p,q)$

The Complete solution is $z = ax + by - 2\sqrt{ab}$

(Replace P = a & q = b)

12. Find the singular integral of $z = px + qy - p^2q^2$

Soln: The Clairaut's form is $z = px + qy + f(p,q)$

The Complete solution is $z = ax + by - a^2b^2$ ……….(i)

Differentiate w.r.t a : 0 = x - b^2

Differentiate w.r.t b : 0 = y - a^2

Eliminate **a** and **b** $\Rightarrow a = \sqrt{y}$, $b = \sqrt{x}$ from (i)

$\therefore z = \sqrt{y}\,x + \sqrt{x}\,y - xy$.

As there is no constant term, that is the singular integral.

13. Find the Complete solution of $p - x^2 = q - y^2$

Soln: Given $p - x^2 = q - y^2$ = a (say)

$\Rightarrow \; p = x^2 + a, q = y^2 + a$.

Now $dz = pdx + qdy \;\Rightarrow\; dz = (x^2 + a)dx + (y^2 + a)dy$

$\int dz = \int (x^2 + a)dx + \int (y^2 + a)dy$

$z = \frac{x^3}{3} + ax + \frac{y^3}{3} + ay + b$. This is Complete solution.

14. Solve $p + q = \sin x + \sin y$ **.**

Soln: Given $p + q = \sin x + \sin y$ = a (say)

$\Rightarrow$ p – sin x = sin y – q = a

$\Rightarrow$ p = sin x +a , q = sin y – a

Now $dz = pdx + qdy \;\Rightarrow\; dz = (\sin x + a)dx + (\sin y - a)dy$

$\therefore \int dz = \int (\sin x + a)dx + \int (\sin y - a)dy$

$z = -\cos x + ax - \cos y - ay + b$. This is complete solution .

15. Find the PDE of the family of spheres having their centre on the line x = y = z.

Soln: The equation of such sphere is

$(x-a)^2 + (y-a)^2 + (z-a)^2 = r^2$ ……. (i)

Partially differentiate w. r. to x and y, we get

$2(x-a) + 2(z-a)p = 0 \;\Rightarrow a = \frac{x + zp}{1+p}$ ….(ii)

$2(y-a) + 2(z-a)q = 0 \;\Rightarrow a = \frac{y + zq}{1+q}$ ….(iii)

From (ii) and (iii), we get $\Rightarrow \frac{x+zp}{1+p} = \frac{y+zq}{1+q}$ which is the required PDE.

16. Solve $(D^3 - 4D^2D' + 4DD'^2)z = 0$

Soln: Given that $(D^3 - 4D^2D' + 4DD'^2)z = 0$ ***(** put $D = m$, $D' = 1$**)**

The Auxiliary equation is: $m^3 - 4m^2 + 4m = 0 \;\Rightarrow$ m = 0, 2, 2

The solution is

$\therefore z = f_1(y + 0.x) + f_2(y + 2x) + x\, f_3(y + 2x)$.

17. Solve $(D^3 + D'D^2 - D^2D' - D'^3)z = 0$

Soln: Given that $(D^3 + D'D^2 - D^2D' - D'^3)z = 0$

***(** put $D = m$, $D' = 1$**)**

The Auxiliary equation is: $m^3 - m^2 + m - 1 = 0$

$\Rightarrow$ m = 1, m = $\pm i$

The solution is $z = f_1(y + x) + f_2(y + ix) + f_3(y - ix)$.

18. Solve $(D - 2D')(D - D')^3 z = 0$.

Soln: Given that $(D - 2D')(D - D')^3 z = 0$

***(** put $D = m$, $D' = 1$**)**

The Auxiliary equation is: $(m-2)(m-1)^3 = 0$

$\therefore$ m=2, m = 1 , 1 , 1

The solution is

$z = f_1(y + 2x) + f_2(y + x) + x\, f_3(y + x) + x^2 f_4(y + x)$.

19. Solve $p \tan x + q \tan y = \tan z$

Soln: $P_p + Q_q = R$ **(Type)**

$\therefore P = \tan x, Q = \tan y, R = \tan z$

The auxiliary eqn. is: $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\Rightarrow \frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$

(i) Consider $\frac{dx}{\tan x} = \frac{dy}{\tan y}$.

Integrating both side we get , $\log \sin x = \log \sin y + \log C_1$

$\therefore \log\left(\dfrac{\sin x}{\sin y}\right) = \log C_1 \Rightarrow \dfrac{\sin x}{\sin y} = C_1$

(ii) Consider $\dfrac{dy}{\tan y} = \dfrac{dz}{\tan z}$.

Integrating we get, $\Rightarrow \dfrac{\sin y}{\sin z} = C_2$

$$\therefore \textit{The Solution is } \left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0 .$$

20. Find the general solution of $px + qy = z$.

Soln: This is the Lagrange's type pf PDE where

$P = x, Q = y, R = z$

The auxiliary equation is: $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R} \Rightarrow \dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dz}{z}$

Consider $\dfrac{dx}{x} = \dfrac{dy}{y} \Rightarrow \dfrac{x}{y} = C_1$

Consider $\dfrac{dy}{y} = \dfrac{dz}{z} \Rightarrow \dfrac{y}{z} = C_2 \quad \therefore \phi\left(\dfrac{x}{y}, \dfrac{y}{z}\right) = 0$ is the solution.

21. Find the Particular integral of $\left(D^2 - 4DD'\right)z = e^{3x+4y}$

Soln: $P.I = \dfrac{1}{D^2 - 4DD'} e^{3x+4y}$

P.I $= \dfrac{1}{9-48} e^{3x+4y} \quad (a = 3, b = 4)$

$= -\dfrac{1}{39} e^{3x+4y}$

22. Form a PDE by eliminating the arbitrary constant a and b from the equation. $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha$

Soln: Given $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha$

Differentiating partially w. r.to x: $2(x-a) = 2z.p\cot^2\alpha$ ……(i)

Differentiating partially w. r.to y: $2(y-b) = 2z.q\cot^2\alpha$ ….(ii)

From (i) and (ii) $\Rightarrow x - a = z.p\cot^2\alpha$ *and* $y - b = z.q\cot^2\alpha$

$\therefore z^2.p^2\cot^4\alpha + z^2.q^2\cot^4\alpha = z^2\cot^2\alpha$

$\Rightarrow p^2 + q^2 = \tan^2\alpha$ Which is the required PDE?

23. Find the singular solution of $z = px + qy + p^2 + q^2 + 1$

Soln: Given $z = px + qy + p^2 + q^2 + 1$

The complete solution is $z = ax + by + a^2 + b^2 + 1$.

put $\dfrac{\partial z}{\partial a} = 0$ *and* $\dfrac{\partial z}{\partial b} = 0 \Rightarrow x + 2a = 0 \ , \ y + 2b = 0$

ie., $a = -\dfrac{x}{2}, b = -\dfrac{y}{2}$

$\therefore z = -\dfrac{x^2}{2} - \dfrac{y^2}{2} + \dfrac{x^2}{4} + \dfrac{y^2}{4} + 1 \quad \Rightarrow 4z = 1 - x^2 - y^2$.

24. Find the complete solution of $\dfrac{z}{pq} = \dfrac{x}{q} + \dfrac{y}{p} + \sqrt{pq}$

Soln : Given that $\dfrac{z}{pq} = \dfrac{x}{q} + \dfrac{y}{p} + \sqrt{pq}$ ………..(i)

From (i) $z = px + qy + (pq)^{3/2}$

The complete solution is $z = ax + by + (ab)^{3/2}$

25. Form the PDE by eliminating arbitrary function from $\phi(z^2 - xy, x/z) = 0$.

Soln: Let $u = z^2 - xy \quad V = \frac{x}{z}$

Then the given eqn is of the form $\phi(u,v) = 0$

The elimination of ϕ from the above eqn we get $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$

$$\begin{vmatrix} 2zp - y & \frac{z - px}{z^2} \\ 2zq - x & \frac{-xq}{z^2} \end{vmatrix} = 0$$

ie., $px^2 - q(xy - 2z^2) = zx$

PART B

Formation of PDE

Form the PDE by eliminating the arbitrary function from the relation

(i) $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$. (ii) $z = x^2 f(y) + y^2 g(x)$.

1. Type I : $F[p,q] = 0$

(i) $\sqrt{p} + \sqrt{q} = 1$ (ii) $pq + p + q = 0$

(iii) $p^2 + q^2 - 4pq = 0$ (iv) $p^2 - 2pq + 3q = 5$

2. **Type II : { Clairut's form }** $z = px + qy + f(p,q)$

(i) $z = px + qy + p^2 - q^2$.

(ii) $z = px + qy + \sqrt{1 + p^2 + q^2}$

(iii) $z = px + qy + -2\sqrt{pq}$

(iv) $\frac{z}{pq} = \frac{x}{q} + \frac{y}{p} + \sqrt{pq}$

(v) $z = px + qy + \sqrt{1 - p^2 - q^2}$

3. Type III : $f(z, p, q) = 0$

(i) $p(1-q^2) = q(1-z)$ (ii) $z = p^2 + q^2$

(iii) $p(1-q^2) = q(1-z)$. (iv) $p(1+q) = qz$

(v) $z^2 = 1 + p^2 + q^2$

4. Type IV : $F_1(x,p) = F_2(y,q)$

(i) $p^2 + q^2 = x^2 + y^2$ (ii) $yp = 2yx + \log q$

(iii) $p^2 y(1 + x^2) = qx^2$

5. Type V : $F(x,y,z,p,q) = 0$

(i) $p^2x^2 + q^2y^2 = z^2$. (ii) $p^2 + q^2 = z^2(x^2 + y^2)$

(iii) $z^2(p^2 + q^2) = x + y$ (v) $px + qy = z$

(vi) $z^2(p^2 + q^2) = x^2 + y^2$.

6. Type: VI Lagrange's method

- This is of the form **:** $Pp + Qq = R$
- The subsidiary Equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(i) $x(y-z)p + y(z-x)q = z(x-y)$

(ii) $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

(iii) $x(z^2-y^2)p + y(x^2-z^2)q = z(y^2-x^2)$

(iv) $(mz-ny)p + (nx-lz)q = ly - mx$

(v) $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

(vi) $(y + z)p + (z + x)q = z + y$

(vii) $x(y^2 + z)p + y(x^2 + z)q = z(x^2 - y^2)$

(viii) $(x + 2z)p + (2xz - y)q = x^2 + y$

(ix) $(y^2 + z^2)p - xyq + xz = 0$

(x) $(z^2 - y^2 - 2yz)p + (xy + zx)q = (xy - zx)$.

7. VII: Homogeneous P.D.E

The Complete solution: z = C.F + P.I

To Find C.F $\Rightarrow$ case (i) If $m_1 = m_2 = m_3$ (real and equal)

$z = f_1(y + m_1x) + x\, f_2(y + m_1x) + x^2\, f_3(y + m_1x)$

case (ii) If $m_1 \neq m_2 \neq m_3$ (real or complex and different)

$z = f_1(y + m_1x) + f_2(y + m_2x) + f_3(y + m_3x)$

To Find P.I $\Rightarrow$

case (i) R.H.S = e^{ax+by}

$\Rightarrow$ P.I = $\frac{e^{ax+by}}{f(D,D')} = \frac{e^{ax+by}}{f(a,b)}$ $\Big\}$ *repalce D by a and D' by b*

Case (ii) R.H.S = $\sin(ax+by)\ (or)\ \cos(ax+by)$

$$\Rightarrow \text{P.I}=\frac{\sin(ax+by)\ (or)\ \cos(ax+by)}{f(D^2,DD',D'^2)}=\frac{\sin(ax+by)\ (or)\ \cos(ax+by)}{f(-a^2,-ab,-b^2)}$$

$\Big\}$ *repalce* D^2 *by* $-a^2$, DD' *by* $-ab$ *and* D'^2 *by* $-b^2$

Case (iii) R.H.S = $x^p y^q$ (p, q being positive integers)

$$\Rightarrow \quad P.I=\frac{1}{f(D,D')}x^p y^q=\left[f(D,D')\right]^{-1}.x^p y^q$$

(i) $(D^3-2D^2D')z=\sin(x+2y)+3x^2y.$

(ii) $(D^2-DD'-6D'^2)z=x^2y+e^{3x+y}.$

(iii) $(D^2-6DD'+5D'^2)z=e^x\sinh y+xy.$

(iv) $(D^2+DD'-6D'^2)z=y\cos x.$

(v) $(D^3-7DD'^2-6D'^3)z=\cos(x+2y)+4.$

(vi) $\left(D^2-DD'-12D'^2\right)z=e^{4x+y}+\sin\left(y+2x\right)+xy$

(vii) $(D^3+D^2D'-DD'^2-D'^3)z=e^x\cos 2y.$

(viii) $(D^2-4D'^2)z=\cos 2x\cos 3y$

(ix) $(D^2+2DD'+D'^2)z=x^2y+e^{x-y}$

(x) $\frac{\partial^2 z}{\partial x^2}-5\frac{\partial^2 z}{\partial x\partial y}+6\frac{\partial^2 z}{\partial y^2}=y\sin x$

(xi) $\left(D^3+D^2D'-DD'^2-D'^3+1\right)z=e^{2x+y}+\cos\left(x+y\right)$

(xii) $(D^2+2DD'+D'^2)z=2\cos y-x\sin y$

(xiii) $(D^2-DD'-20D'^2)z=e^{(5x+y)}+\sin(4x-y).$

(xiv) $(D^2-6DD'+9D'^2)z=12x^2+36xy.$

(xv) $(D^2-DD'-2D'^2)z=2x+3y+e^{2x+4y}$

(xvi) Solve $(D^2-3DD'+2D'^2)z=(2+4x)e^{x+2y}.$

(xvii) Solve $(D^2+3DD'+2D'^2)z=\sin(x+5y).$

(xviii)Solve $(D^3-2D^2D')z=2e^{2x}+3x^2y$

(xix) Solve $(D^3-7DD'^2-6D'^3)z=\sin(x+2y).$

Non-Homogeneous P.D.E

(i) Solve $(D-D'-1)(D-D'-2)z=e^{2x-y}.$

(ii) Solve $(2DD'+D'^2-3D')z=3\cos(3x-2y).$

(iii) Solve $(D^2+2DD'+D'^2-2D-2D')z=\sin(x+2y).$

(iv) Solve $(D^2+3DD'+D'^2-2D-2D')z=\sin(2x+y).$

8. Solve $\frac{\partial^2 z}{\partial x^2}=a^2z$ given that when $x=0,\ \frac{\partial z}{\partial x}=a\sin y$ and $\frac{\partial z}{\partial y}=0.$

Unit II
Fourier series

1. Write down the Dirichlet's conditions for a function to be expanded as a Fourier series.

Solution:

i) f(x) is single-valued, periodic and finite in (c,c+2π).

ii)f(x) has finite number of finite discontinuity and no infinite discontinuity in (c,c+2π).

iii) f(x) has finite number of maxima and minima in (c,c+2π).

2. What is the constant term and coefficient of $\cos nx$, a_n in the Fourier series? Expansion of $f(x)=x-x^3$ in $(-7,7)$?

Solution:

Given f(x)=x-x^3 and since f(-x)= -x+x^3=-(x-x^3)=-f(x).The given function is an odd function,a_0 and a_n is zero.

3. If the cosine series for f(x) = xsinx ,0< x< π is given by

xsinx = $1-\frac{1}{2}\cos x-2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx$, **show that**

$1+2\left[\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-...\right]=\frac{\pi}{2}$.

Soln: Put x = $\frac{\pi}{2}$, which is a point of continuity.

f(x)= f($\frac{\pi}{2}$) = $\frac{\pi}{2}$sin($\frac{\pi}{2}$) = $\frac{\pi}{2}$.

Sub x = $\frac{\pi}{2}$ in the fourier series.

$$1-\frac{1}{2}\cos\frac{\pi}{2}-2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos n\frac{\pi}{2}=\frac{\pi}{2}.$$

$$1+2\left[\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-...\right]=\frac{\pi}{2}.$$

4. If the Fourier series of the function $f(x)=x+x^2 \ in -\pi\le x\le\pi$

is $\frac{\pi^2}{3}+\sum_{1}^{\infty}(-1)^n\left\{\frac{4\cos nx}{n^2}-\frac{2\sin nx}{n}\right\}$**, then find value of the infinite**

series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$

Solution:

Given f(x) =x+x² = $\frac{\pi^2}{3}+\sum_{1}^{\infty}(-1)^n\left\{\frac{4\cos nx}{n^2}-\frac{2\sin nx}{n}\right\}$

Put x= π, (point of discontinuity), we get,

$$\frac{f(\pi)+f(-\pi)}{2}=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}=\frac{\pi+\pi^2+\pi^2-\pi}{2}$$

$$=\pi^2\Rightarrow\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}\Rightarrow\sum_{n=1}^{\infty}\frac{4}{n^2}=\pi^2-\frac{\pi^2}{3}\Rightarrow\sum_{n=1}^{\infty}\frac{4}{n^2}=\frac{2\pi^2}{3}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

Hence the result.

5. If the Fourier series corresponding to $f(x)=x$ **in the interval** $(0,2\pi)$ **is** $\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$**. Without finding the values of** a_o,a_n,b_n **Find the value of** $\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$**.**

Solution:

By using Parsevals identity

$$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)=\frac{1}{\pi}\int_0^{2\pi}x^2dx=\frac{1}{\pi}(\frac{x^3}{3})_0^{2\pi}=\frac{1}{3\pi}\times8\pi^3=\frac{8\pi^2}{3}$$

6. Find the constant term in the Fourier series corresponding to $f(x)=\cos^2 x$ **expressed in the interval** $(-\pi,\pi)$**?**

Solution:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1+\cos 2x}{2}dx=\frac{1}{\pi}[x+[\frac{\sin 2x}{2}]_0^{\pi}=1$$

7. If $f(x)$ **is defined in** $-3\le x\le 3$**, what is the value of** b_n **?**

Solution:

$$b_n=\frac{1}{3}\int_{-3}^{3}f(x)\sin\frac{n\pi x}{3}dx$$

8 .Find a_0 **in expanding** e^x **as Fourier series in** $(-\pi,\pi)$ **?**

Solution:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x dx \Rightarrow \frac{1}{\pi}[e^x]_{-\pi}^{\pi} \Rightarrow \frac{1}{\pi}[e^{\pi} - e^{-\pi}]$$

$$= \frac{2\sinh \pi}{\pi}$$

9. f(x)= $\begin{cases} 1-\dfrac{2x}{\pi}, -\pi < x < 0 \\ 1+\dfrac{2x}{\pi}, 0 < x < \pi \end{cases}$ **is even. State true or false**.

Answer : True.

f(-x)= $\begin{cases} 1+\dfrac{2x}{\pi}, -\pi < -x < 0 \\ 1-\dfrac{2x}{\pi}, 0 < -x < \pi \end{cases}$

$= \begin{cases} 1-\dfrac{2x}{\pi}, -\pi < x < 0 \\ 1+\dfrac{2x}{\pi}, 0 < x < \pi \end{cases}$

= f(x).

10. $x = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} ...\right)$ **in** $0 < x < \pi$ **. Prove that**

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Solution:

Given the sine series expansion of f(x) =x & from this $b_n = \frac{2(-1)^{n+1}}{n}$

By Parseval's identity,

$$\bar{y}^2 = \frac{1}{2}\sum b_n^{\ 2}$$

$$\frac{1}{\pi}\int_0^{\pi} x^2 dx = 2\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{3\times 2} = \frac{\pi^2}{6}$$

11. State Parseval's Identity for the Half –Range COSINE expansion of $f(x)$ **in** $(0,l)$ **.**

Solution:

$$2\int_0^l [f(x)]^2 dx = \frac{a_0^{\ 2}}{2} + \sum_{n=1}^{\infty} a_n^{\ 2}$$

$$where\ a_0 = 2\int_0^l f(x)dx$$

$$a_n = 2\int_0^l f(x)\cos nx\, dx$$

12. State Parseval's Identity of Fourier series.

Solution: If f(x) has a Fourier series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\, in\, (0, 2\pi), then$$

$$\frac{1}{2\pi}\int_0^{2\pi} [f(x)]^2 dx = \frac{a_0^{\ 2}}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^{\ 2} + b_n^{\ 2})$$

13. What do you mean by Harmonic analysis?

The process of finding the fourier series for a function given by numerical value is known as harmonic analysis.

In harmonic analysis the Fourier coefficients a_o, a_n, b_n of the function y = f(x) in $(0, 2\pi)$ are given by

a_0 = 2[mean value of y in $(0, 2\pi)$]

a_n = 2[mean value of ycosnx in $(0, 2\pi)$]

bn = 2[mean value of ysinnx in $(0, 2\pi)$]

14. **Define R.M.S value of a function** $f(x)$ **in** $a < x < b$ **.**

R.M.S value of a function $f(x)$ in $a < x < b$ is defined by

$$\bar{y}^2 = \frac{1}{b-a}\int_a^b [f(x)]^2 dx$$

15. Find the R.M.S value of f(x) = x in 0 < x < l **.**

$$\bar{y}^2 = \frac{1}{l}\int_0^l x^2 dx = \frac{l^2}{3}.$$

$\bar{y} = \frac{l}{\sqrt{3}}$.

16. Find the R.M.S value of $y = x^2$ in $(-\pi, \pi)$?

Solution: If $\bar{y}$ is the R.M.S value, then

$$\bar{y}^2 = \frac{1}{\pi-(-\pi)}\int_{-\pi}^{\pi} x^4 dx \Rightarrow \frac{2}{2\pi}\int_0^{\pi} x^4 dx \Rightarrow \frac{1}{\pi}[\frac{x^5}{5}]_0^{\pi}$$

$$\bar{y}^2 = \frac{\pi^4}{5}, \therefore \bar{y}^2 = \frac{\pi^2}{\sqrt{5}}$$

17. Find the Half –Range SINE series of $f(x) = x$ in $(0, \pi)$?

Solution:

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx dx \Rightarrow [-x(\frac{\cos nx}{n}) + \frac{\sin nx}{n^2}]_0^{\pi}$$

$$= \frac{2}{\pi}[-\pi\frac{\cos n\pi}{n}] \Rightarrow \frac{2(-1)^{n+1}}{n}$$

The required sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = 2[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - ....]$$

18. Find the value of a $_n$ in the cosine series expansion of f (x) = k in the interval (0,10).

$$a_n = \frac{2}{10}\int_0^{10} k \cos\frac{n\pi x}{10} dx = \frac{2}{10}k\left(\frac{\sin\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)_0^{10} = 0.$$

19. Find b_n in the expansion of x^2 as a Fourier series in $(-\pi, \pi)$

Solution:

b_n=0, since f(x) is an even function in $(-\pi, \pi)$.

20. If f(x) is an odd function defined in (-l,l).what are the values of a_0 and a_n?

Solution:

a_0=a_n=0, since f(x) is an odd function.

21. Find a Fourier series for the function f(x) =1; 0<x< π .

Solution: The fourier series of

$$f(x) = \sum_{n=1}^{\infty} bn \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx \Rightarrow \frac{2}{\pi}\int_0^{\pi} 1 \sin nx dx$$

$$= \frac{2}{\pi}[\frac{-\cos nx}{n}]_0^{\pi} = \frac{-2}{n\pi}[(-1)^n - 1]$$

$$b_n = \begin{cases} 0, \text{when } n \text{ is even} \\ \frac{4}{n\pi}, \text{when } n \text{ is odd} \end{cases}$$

$$\therefore f(x) = \sum_{n=1,3,5}^{\infty} \frac{4}{n\pi}\sin nx$$

22. If the fourier series for the function f(x) =0, 0<x< π is

$$f(x) = \frac{-1}{\pi} + \frac{2}{\pi}[\frac{\cos 2x}{1.3} + \frac{\cos 4x}{3.5} + ...] + \frac{1}{2}\sin x. \text{deduce that } \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - .....\infty = \frac{\pi - 2}{4}$$

Solution: Put, $x = \frac{\pi}{2}$, we get

$$f(\frac{\pi}{2}) = \frac{-1}{\pi} + \frac{2}{\pi}[-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + ...\infty] + \frac{1}{2}$$

$$0 = -\frac{1}{\pi} - \frac{2}{\pi}[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - ......\infty] + \frac{1}{2}$$

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - ......\infty = (\frac{1}{\pi} - \frac{1}{2})(-\frac{\pi}{2})$$

$$= \frac{\pi - 2}{4}$$

23. Write the Fourier sine series of k in (0, π)

Solution:

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx \Rightarrow \frac{2}{\pi}\int_0^{\pi} k\sin nx dx=\frac{2k}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}=\frac{2k}{n\pi}[1-(-1)^n]$$

$$b_n=\begin{cases}0, n \ \ is\ even\\ \frac{4k}{n\pi}\end{cases}$$

$$\therefore f(x)=\sum_{n=1,3,5..}^{\infty}\frac{4k}{n\pi}\sin nx \Rightarrow \frac{4k}{\pi}\sum_{n=1,3,5..}^{\infty}\frac{1}{2n-1}\sin(2n-1)x.$$

24.Write the complex form of Fourier series for f(x) defined in the interval c, c+2*l*.

Ans: $$f(x)=\sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$$

Where $$C_n=\frac{1}{2l}\int_c^{c+2l} f(x)e^{\frac{-in\pi x}{l}}dx$$

25.If $f(x)=|x|$ expanded as a Fourier series in $-\pi<x<\pi$ Find a_0.

Ans: $$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}|x|dx$$

$$=\frac{2}{\pi}\int_0^{\pi} xdx \ \ [\because |x| \text{ is an even function}]$$

$$=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}=\pi$$

PART B

1. Express $f(x)=(\pi-x)^2$ as a Fourier series of period 2π in the interval $0<x<2\pi$.Hence deduce the sum of the series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+.......$
2. Obtain the Fourier series of $f(x)=kx(\pi-x)$ in $(0,2\pi)$.
3. Obtain the Fourier series of $f(x)=x^2$ in $(0,2\pi)$.
4. If $f(x)=\frac{1}{2}(\pi-x)$, find the Fourier series of period 2π in $(0,2\pi)$.

   Hence deduce that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+....=\frac{\pi}{4}$.

5. Find the Fourier series expansion of $f(x)=\begin{cases}1 & :0\le x\le \pi\\ 2 & :\pi\le x\le 2\pi\end{cases}$
6. Find the Fourier series expansion of period $2l$ for the function $f(x)=(l-x)^2$ in the range $(0,2l)$.Deduce the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

7. Find a Fourier series expansion for $f(x)=\begin{cases}x & :0\le x\le 3\\ 6-x & :3\le x\le 6\end{cases}$

8.Find the Fourier series expansion of $f(x)=\begin{cases}2-x & :-2\le x\le 0\\ 2+x & :0<x\le 2\end{cases}$.

Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+....=\frac{\pi^2}{8}$.

9. Find the Fourier series of $f(x)=\begin{cases}l-x, & 0\le x\le l\\ 0, & l\le x\le 2l\end{cases}$. Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+....=\frac{\pi^2}{8}$ and $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+....=\frac{\pi}{4}$.

10. Determine the Fourier series expansion of $x\sin x$ in $-\pi \le x \le \pi$.

11. Find the Fourier series expansion of $f(x)=\begin{cases}\pi + x, & -\pi \le x \le 0\\ \pi - x, & 0 \le x \le \pi\end{cases}$.

Hence deduce that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$.

12. Determine the Fourier expansion of $f(x)=|x|, -\pi < x < \pi$.

13. Obtain the Fourier expansion for $f(x)=|\sin x|$ in the interval $-\pi < x < \pi$.

14. Obtain the Fourier expansion for $f(x)=|\cos x|$ in the interval $-\pi < x < \pi$.

15. Obtain the Fourier series expansion of

$$f(x)=\begin{cases}1+\frac{2x}{\pi}, & -\pi \le x \le 0\\ 1-\frac{2x}{\pi}, & 0 \le x \le \pi\end{cases}.$$ Deduce

that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+.......=\frac{\pi^2}{8}$

16. Find the Fourier series of $f(x)=\begin{cases}0, & -\pi \le x \le 0\\ \sin x, & 0 \le x \le \pi\end{cases}$. Hence

show that

(i) $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+....=\frac{1}{2}$; (ii) $\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-....=\frac{\pi-2}{4}$.

17. Find the Fourier series expansion for $f(x)=x^2$ in $(-\pi,\pi)$

and hence show that $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\infty=\frac{\pi^4}{90}$.

18. Find the Fourier series expansion for $f(x)=(\pi-x)^2, 0<x<\pi$.

Hence deduce that $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\infty=\frac{\pi^4}{90}$.

19. Obtain the Fourier series of period 2π, for the function $f(x)=x^2$ in $(-\pi,\pi)$. Deduce the result

$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+.......=\frac{\pi^2}{6}$ and $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-.......=\frac{\pi^2}{12}$ and

$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+.......=\frac{\pi^2}{8}$.

18. Find the Fourier series expansion of $f(x)=\begin{cases}-x+1 & :-\pi \le x \le 0\\ x+1 & :0 \le x \le \pi\end{cases}$

19. Find the Complex form of the Fourier series $f(x)=\cos ax$ in $-\pi < x < \pi$.

20. Find the Complex form of the Fourier series of $f(x)=e^{ax}$ in $(-l,l)$ and in $(-\pi,\pi)$.

21. Find the Complex form of the Fourier series f(x) = sin x, $-\pi < x < \pi$.

22. Expand $f(x)=e^{-x}$ as F.S in (-1,1).

23. Derive the complex form of Fourier series for $f(x)=e^{ax}$, $-\pi < x < \pi$. Given that 'a' is real constant, Deduce that

(i) $e^{ax}=\frac{\sinh a\pi}{\pi}\sum_{-\infty}^{\infty}(-1)^n\frac{a+in}{a^2+n^2}e^{inx}$

(ii) $\cos ax=\frac{\sin\pi a}{\pi}\sum_{-\infty}^{\infty}(-1)^n\frac{a}{a^2-n^2}e^{inx}$

(iii) $\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2+a^2}=\frac{\pi}{a\sinh a\pi}$.

24. Find half range cosine series for $f(x)=x$ in $[0,\pi]$ and $[0,l]$. Deduce the sum of the series $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+......$

25. Find half range cosine series for $f(x) = x(2-x)$ in $0 \le x \le 2$.

Deduce the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$

26. Find the half range sine & cosine series for

$f(x) = x(\pi - x)$ in $(0,\pi)$. Deduce that $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \ldots\ldots = \frac{\pi^3}{32}$.

27. Obtain sine &cosine series for the function $f(x) = \begin{cases} x & : 0 \le x \le \frac{l}{2} \\ l - x : & \frac{l}{2} \le x \le l \end{cases}$

28. Find the half range sine series for

f(x) = 4x - x ^{2} (0,4). Deduce that $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \ldots\ldots\infty$

29. Find the half range sine series for $f(x) = lx - x^2 \quad in\ (0,l)$

30. Find the Fourier series upto third harmonic for the function y=f(x) defined in $(0,\pi)$ from the table.

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π
$f(x)$	2.34	2.2	1.6	0.83	0.51	0.88	1.19

31. Find the F.S up to the third harmonic for $y = f(x)$ in $(0, 2\pi)$ defined by the table of values given below.

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
$f(x)$	1	1.4	1.9	1.7	1.5	1.2	1.0

32. Determine the first two harmonics of Fourier series for the following data .

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$
$f(x)$	1.98	1.30	1.05	1.30	-0.88	-0.25

33. The following table gives the variations of a periodic function over a period T.

x	0	$\frac{T}{6}$	$\frac{T}{3}$	$\frac{T}{2}$	$\frac{2T}{3}$	$\frac{5T}{6}$	T
$f(x)$	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Show that $f(x) = 0.75 + 0.37\cos\theta + 1.004\sin\theta$, where $\theta = \frac{2\pi x}{T}$.

34. Find the constant term and the coefficients of the first sine and cosine terms in the Fourier expansion of y as given in the following

x	0	1	2	3	4	5
y	9	18	24	28	26	20

35. Find the Fourier cosine series up to third harmonic to represent the function given by the following data:

x	0	1	2	3	4	5
y	4	8	15	7	6	2

Unit III
Applications of Partial Differential Equations

1. **What is the basic difference between the solutions of ODWE & ODHE?**

Soln : Solution of the ODWE is of periodic in nature , but solution of the ODHE is not of periodic in nature .

2. **In steady state conditions derive the solutions of one dimensions heat flow equation.**

Soln: When steady state conditions exist the heat flow equation is independent of time't'. $\frac{\partial u}{\partial t} = 0$

The heat flow equation becomes $\frac{\partial^2 u}{\partial x^2} = 0$.

3. **In the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$. What does c^2 stand for?**

Soln: $c^2 = \frac{T}{m} = \frac{Tension}{mass}$.

4. **State ODHE with the initial and boundary conditions .**

Soln: The ODHE is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$.

The boundary conditions are $a)\ u(0,t) = k_1^\circ c \ \ for\ all\ t > 0$

$b)\ u(l,t) = k_2^\circ c \ \ for\ all\ t > 0$

The initial condition is $u(x,0) = f(x), 0 < x < l$.

5. **State Fourier law of heat condition.**

Soln: The rate at which heat flows across an area A at a distance x from one end of a bare is given

By $Q = -k\,A\left(\frac{\partial u}{\partial x}\right)_x$ K is thermal conductivity and

$\left(\frac{\partial u}{\partial x}\right)_x$ the temperature gradient at x.

6. Classify the PDE $u_{xx} + x u_{yy} = 0$.

Soln: Here A = 1, B = x, c = 0 $\Rightarrow B^2 - 4AC = x^2$

i) Elliptic if x >0 ii) Parabolic if x = 0 iii) Hyperbolic if x < 0.

7. Classify the following second order PDE.

$a)\ 4\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} - 6\frac{\partial u}{\partial x} - 8\frac{\partial u}{\partial y} - 16u = 0$

$b)\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$

Soln: a) A = 4, B = 4, c =1. $\therefore B^2 - 4AC = 0 \Rightarrow$ It is parabolic equation.

b) A =1, B = 0, c =1 $\therefore B^2 - 4AC = -4 < 0 \Rightarrow$ It is Elliptic equation.

8. A rod 30 cm long has its ends A and B kept at $20^\circ c$ *and* 80° respectively until steady conditions prevail. Find the steady state temp in the rod.

Soln: Let $l = 30\,cm$, when steady condition prevail the heat flow equation is $\frac{\partial^2 u}{\partial x^2} = 0$. i.e., u (x) = a x +b…….. (i)

When steady state conditions exists, the B.C's are

$u(0) = 20; u(l) = 80$ ……….(ii)

Apply (ii) in (i) u(0) = b = 20 …….(iii) and u(l)= al +20=80

$\therefore a = \frac{60}{l}$ ……(iv)

Substitute (iii) and (iv) in (i) $\Rightarrow u(x) = \frac{60x}{l} + 20$.

9. Classify the PDE $\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$.

Soln: Given that $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ **,** Here A **=** α^2 **,** B = 0 **,** C =0

$\Rightarrow \therefore B^2 - 4AC = 0 \Rightarrow$ It is parabolic equation.

10. Explain the initial and boundary value problem.

Soln: IVP: In ordinary differential equation the general solution contain the arbitrary constants; we determine these constants from the given initial values which is known as initial value problem.

BVP: In ordinary or PDE the solution of the differential equation satisfies some specified condition called Boundary condition, any differential equation together with these boundary conditions is called BVP.

11. Define Temperature Gradient.

Soln: The rate of change of temperature with respect to distance is called temperature and is denoted

By $\frac{\partial u}{\partial x}$. We note that $\frac{\partial u}{\partial x}$ = limiting value of $\frac{u_1 - u_2}{x}$.

12. Write down the three possible solution of ODHE.

Soln: $i)\ u(x,t)=(c_1x+c_2)c_3$

$ii)\, u(x,t)=(c_4\, e^{px}+c_5\, e^{-px})\, c_6\ e^{\alpha^2 p^2 t}$

$iii)\, u(x,t)=(c_7\cos px+c_8\sin px)\quad c_9\ e^{-\alpha^2 p^2 t}$.

13. State all the possible (or) various solutions of one dimensional wave equation.

Soln: $(i)\ y=(c_1x+c_2)(c_3t+c_4)$

$(ii)\, y=\left(c_1\, e^{px}+c_2\, e^{-px}\right)\left(c_3\, e^{pct}+c_4\, e^{-pct}\right)$

$(iii)\, y=(c_1\cos px+c_2\sin px)(c_3\cos pct+c_4\sin pct)$

14. State all the possible (or) various solutions of Laplace equation obtained by the method of separation of variables.

Soln: $(i)\ y=(c_1x+c_2)(c_3y+c_4)$

$(ii)\, y=\left(c_1\, e^{px}+c_2\, e^{-px}\right)(c_3\cos py+c_4\sin py)$

$(iii)\, y=(c_1\cos px+c_2\sin px)\left(c_3\, e^{py}+c_4 e^{-py}\right)$

15. A tightly stretched string with fixed end points $x=0$ *and* $x=l$ **is initially at rest is its equilibrium position. It is set vibrating giving each point a velocity** $y_o\sin^3\left(\frac{\pi x}{l}\right)$**. Write down Initial condition and Boundary condition.**

Soln: Boundary condition: $i)\, y(0,t)=0\quad ii)\ y(l,t)=0$

Initial condition:

$i)\, y(x,0)=0\quad ii)\frac{\partial}{\partial t}y(x,0)=y_o\sin^3\left(\frac{\pi x}{l}\right),\ 0<x<l$.

16. Write the initial condition of the wave equation if the string has an initial velocity.

Soln: The initial conditions are $i)\, y(x,0)=0\ \ for,\ 0<x<l$

$ii)\frac{\partial}{\partial t}y(x,0)=f(x),\ 0<x<l$.

17. State thermally insulated ends.

Soln: It there will be not heat flow passes through the end of the bar then that two ends are said to be thermally insulated.

18. Write the boundary conditions and initial conditions for solving the vibration of string equation, if the string is subjected to initial displacement f(x) and initial velocity g(x).

Soln: The wave equation is $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$. The B.C's and I.C's are

$i)\, y(0,t)=0\quad ii)\ y(l,t)=0\ \ iii)\ \frac{\partial}{\partial t}y(x,0)=g(x)\quad iv)\, y(x,0)=f(x)$.

19. Classify the PDE $A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$ **, where A,B,C,D,E,D,E,F,G are functions of x and y only.**

Soln: 1. Elliptic if $B^2-4AC<0$

2. Parabolic $B^2-4AC\ =0$

3. Hyperbola $B^2-4AC>0$.

20. Classify the PDE

$x^2 f_{xx}+\left(1-y^2\right)f_{yy}=0,\ -\infty<x<\infty\ and\ -1<y<1$

Soln: Here A = x^2 , B = 0, C = $1-y^2$ $\quad\therefore B^2-4AC=0-4x^2\left(1-y^2\right)$

$for\ \ -1<y<1,\ 1-y^2>0\ \ and\ \ x^2>0\ always.$

$\therefore B^2-4AC<0\ \Rightarrow The\, equation\ is\ elliptic$.

21. In the one dimensional heat equation $\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial t^2}$, *what is* c^2?

Soln:
$c^2 = \frac{K}{\rho . S}$ *where K = thermal conductivity, ρ = density,*
S = *specific heat* .

22. Write down the two dimensional heat equations in transient state.

Soln: $\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$

23. Write down the two dimensional heat equations in steady –state in Cartesian form.

Soln: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

24. Write down the general solution for he string problem when velocity is given.

Soln: $y = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right).\sin\left(\frac{n\pi ct}{l}\right).$

25. What are Transient and Steady state?

Soln: The state in which temperature on a bar / rod changes with change of time is called **Transient State.**

The state in which temperature on a bar / rod remains constant with change of time is called Steady **state**.

26. Write down the Two Dimensional Heat Equation.

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (or) $u_{xx} + u_{tt} = 0$ (or) $\nabla^2 u = 0$ (Laplace Equation)

27. Give the suitable solution of Two Dimensional Heat Equation.

i. $u(x, y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py})$

ii. $u(x, y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py)$

28. What is the main difference between steady state and unsteady state.

i. Temperature depends on both position and time in Steady State.

ii. Temperature depends only on the position not time in Unsteady State.

29. Give the D'Alemberts Solution when the string is at rest.

$y = \frac{1}{2}[\varphi(x + ct) + \varphi(x - ct)]$

PART B

(Separation of variables)

1. Solve using separation of variables method

   (i) $yu_x + xu_y = 0$ (ii) $x^2q + y^3p = 0$

   (iii) $u_x = 2u_t + u$, where $u(x,0) = 6e^{-3x}$

2. Obtain the solution of wave equation.

O. D. W. E (Vibration of String - Initial velocity is zero)

1. A string is stretched and fastened to two points x = 0 and x = l apart. Motion is started by displacing the string into the form $y = k\left(lx - x^2\right)$ from which it is released at time $t = 0$. Find the displacement of any point on the string at a distance of x from one end at time t.
2. A tightly stretched string with fixed end points $x = 0$ *and* $x = l$ is initially in a position given by $y(x,0) = y_o \sin^3 \frac{\pi x}{l}$. If it is released from rest from this position find the displacement y at any distance x from one end at any time t.
3. A string is tightly stretched and its ends are fastened at two points $x = 0$ *and* $x = l$. The midpoint of the string is displaced transversely through a small distance $'b'$ and the string is released from rest in that position. Find the expression for the transverse displacement of the string at any time during the subsequent motion.
4. A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = A\sin\frac{\pi x}{l}$ from which it is released at time t=0. Show that the displacement of any point at a distance x from one end at time t is given by $y(x,t) = A\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}$.

(Vibration of String - Initial velocity is given)

1. If a string of length $'l'$ is initially at rest its equilibrium position and each of its points is given the velocity is $y_o \sin^3 \frac{\pi x}{l}$; $0 < x < l$.
   Determine the displacement function y (x, t).
2. A tightly stretched string with fixed end points $x = 0$ *and* $x = l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $\lambda x(l - x)$. Determine the displacement function y (x, t).
3. A string is stretched between two fixed points at a distance 2l apart and the points of the string are given initial velocities v given by v = $\frac{cx}{l}$ in 0< x< l

   = $\frac{c(2l-x)}{l}$ in l<x< 2l, x being the distance from one end point. Find the displacement of the string at any subsequent time.
4. If a string of length l is initially at rest in its equilibrium position and each of its points is given a velocity v such that v

   = 2kx/l, 0<x< l/2

   = 2k(l-x)/l, l/2<x<l

   Determine the displacement function y(x,t) at any time t
5. A tightly stretched string of length 2l is fastened at x = 0 and x = 2l. The midpoint of the string is then taken to height ‘b’ transversely and then released from rest in that position. Find the lateral displacement of the string.
6. A tightly stretched string of length l with with fixed end points is initially at rest in its equilibrium position. If it is set vibrating by giving each point a velocity $y_t(x,0) = v_0 \sin^3\left(\frac{3\pi x}{l}\right)\cos\left(\frac{\pi x}{l}\right)$, where 0<x<l. Find the displacement of the string at a point, at a distance x from one end at any instant ‘t’.

ONE & TWO DIMENSIONAL HEAT FLOW EQUATION (OR) LAPLACE EQUATION:

1. A square plate is bounded by the lines $x = 0, y = 0, x = 20$ *and* $y = 20$. Its faces are insulated. The

temperature along the upper horizontal edge is given by $u(x,20) = x(20-x)$ *when* $0 < x < 20$ while the other three edges are kept at $0^\circ c$. Find the steady state temperature in the plate.

2. A rod 30cm long has its ends A and B kept at 20° and 80° respectively until steady state conditions prevail the temperature at each end is then suddenly reduced to $0^\circ C$ and kept so. Find the resulting temperature function $u(x,t)$ taking x=0 at A.
3. The ends A and B of a rod 30cms long have their temperature kept at $20^\circ C$ and the other at 80° C until steady state conditions prevail. The temperature of the end B is then suddenly reduced to $60^\circ C$ and kept so while the end A is raised to $40^\circ C$. Find the temperature distribution in the rod after time t.
4. An infinitely long rectangular plate with insulated surfaces is 10 cm wide. The two long edges and one short edges are kept at zero temperature, while the other short edge x = 0is kept at temperature given by $u = \begin{cases} 20y & ;\ for\ 0 \le y \le 5 \\ 20(10-y) & ;\ for\ 5 \le y \le 10 \end{cases}$. Find the steady state temperature distribution in the plate.
5. Solve the equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ subject to the boundary conditions $u(0,t) = 0,\ u(l,t) = 0\ and\ u(x,0) = x.$
6. Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with boundary conditions $u(0,t) = 0,\ u(l,t) = 0\ and\ u(x,0) = 3\sin n\pi x.$ where $0 < x < l.$
7. A rectangular plate with insulated surfaces is 20 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature while the other short edge x = 0 is give by $u = \begin{cases} 10y & for\ 0 \le y \le 10 \\ 10(20-y) & for\ 10 \le y \le 20 \end{cases}$ and the two long edges as well as the other short edge are kept at zero temperature, find the steady state temperature distribution u(x,y) in the plate.

Unit IV
Fourier Transform

1. State Fourier integral theorem.

Solution:

If f(x) is piecewise continuously differentiable and absolutely integrable in $(-\infty,\infty)$, then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{is(x-t)}dtds\ \ OR\ \ f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dtd\lambda$$

2. Define Fourier transform pair.

Solution:

If f(x) is defined in $(-\infty,\infty)$,then its FT is defined by

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x).e^{isx}dx..........(1)$$

If F(s) is the FT of f(x), then every point of continuity f(x), we have

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}dx........(2)$$

Expressions (1) & (2) are called FT pair.

3. Find the Fourier transform of $f(x)$ is defined by $f(x) = \begin{cases} 1\ ;\ a < x < b \\ 0\ ;otehwise \end{cases}$

Solution:

The FT is

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{isx} dx = \frac{1}{\sqrt{2\pi}} [\frac{e^{isx}}{is}]_a^b = \frac{1}{\sqrt{2\pi}} [\frac{e^{isb} - e^{isa}}{is}]$$

4. Find the Fourier transform of $e^{-|x|}$.

Solution: $F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{isx} dx \ [\because e^{isx} = \cos sx + i \sin sx]$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} \cos sx\, dx + \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} \sin sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-x} \cos sx\, dx = \sqrt{\frac{2}{\pi}} [\frac{1}{1+s^2}] \ \ [\because e^{-ax} \cos bx dx = \frac{a}{a^2+b^2}]$$

5. Find FT of $f(x) = \begin{cases} 1 \ ; |x| < a \\ 0 \ ; |x| > a > 0 \end{cases}$

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx} dx = \frac{1}{\sqrt{2\pi}} [\frac{e^{isx}}{is}]_{-a}^{a} = \frac{1}{s}\sqrt{\frac{2}{\pi}} [\frac{e^{isa} - e^{-isa}}{2i}] = \sqrt{\frac{2}{\pi}} (\frac{\sin sa}{s})$$

6. If F(s) is the FT of f(x), then find the FT of f(x-a)
Solution:

$$F(f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} dx$$

Put x-a = t, dx = dt

$x \to -\infty, t \to -\infty$

$x \to \infty, \ t \to \infty$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).e^{is(t+a)} dt = \frac{e^{ias}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).e^{ist} dt = \frac{e^{ias}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x).e^{isx} dx = e^{ias} F(s)$$

7. Solve the integral equation $\int_0^{\infty} f(x) \cos \lambda x.dx = e^{-\lambda}$

Solution:

Given $\int_0^{\infty} f(x) \cos \lambda x.dx = e^{-\lambda}$

replace λ *by* s, *weget* $\int_0^{\infty} f(x) \cos sx.dx = e^{-s}$.......(1)

By definition of FCT we have

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx.dx$$

$$\int_0^{\infty} f(x) \cos sx.dx = \sqrt{\frac{\pi}{2}} F_c[f(x)] ......(2)$$

from (1) & (2) we get

$$\sqrt{\frac{\pi}{2}} F_c[f(x)] = e^{-s}$$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} e^{-s} ,$$

Using inverse formula,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c[f(x)] \cos s\, dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} e^{-s} \cos sx\, dx$$

$$= \frac{2}{\pi} \int_0^{\infty} e^{-s} \cos sx\, dx = \frac{2}{\pi} [\frac{1}{x^2+1}]$$

8. Find FST of $f(x) = e^{-x}$

Solution:

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x).\sin sxdx = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}.\sin sxdx$$

$$= \sqrt{\frac{2}{\pi}}[\frac{e^{-x}}{s^2+1}[-\sin sx - s\cos sx]]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}[0 - \frac{e^0}{s^2+1}(-s\cos 0)] = \sqrt{\frac{2}{\pi}}(\frac{s}{s^2+1})$$

9. Find FCT of $f(x) = e^{-x}$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x).\cos sxdx = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}.\cos sxdx$$

$$= \sqrt{\frac{2}{\pi}}[\frac{e^{-x}}{s^2+1}[-\cos sx + s\sin sx]]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}(\frac{1}{s^2+1})$$

10. Find FST of $\frac{1}{x}$

Solution:

We Know that

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin sx}{x}dx = \sqrt{\frac{2}{\pi}} \times \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$$

11. Find FCT of $f(x) = \begin{cases} x & ;0 < x < 1 \\ 2-x & ;1 < x < 2 \\ 0 & ;x > 2 \end{cases}$

Solution:

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^1 x\cos xdx + \sqrt{\frac{2}{\pi}}\int_1^2 (2-x)\cos xdx$$

$$= \sqrt{\frac{2}{\pi}}[(x\frac{\sin sx}{s})]_0^1 - \int_0^1 \frac{\sin sx}{s}dx + [(2-x)(\frac{\sin sx}{s})]_1^2 + \int_1^2 \frac{\sin sx}{s}dx$$

$$= \sqrt{\frac{2}{\pi}}[\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{\sin s}{s} - \frac{\cos 2s}{s^2} + \frac{\cos s}{s^2} - \frac{\cos 0}{s^2}]$$

$$= \sqrt{\frac{2}{\pi}}[\frac{2\cos s - \cos(2s-1)}{s^2}]$$

12. F.C.T of $f(x) = \begin{cases} x \ ; 0 < x < \pi \\ 0 \ ; \quad x \ge \pi \end{cases}$

Solution:

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\pi x\cos xdx$$

$$= \sqrt{\frac{2}{\pi}}\left[\left[(x\frac{\sin sx}{s})\right]_0^\pi - \int_0^\pi \frac{\sin sx}{s}dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[0 + \left[\frac{\cos sx}{s^2}\right]_0^\pi\right] = \sqrt{\frac{2}{\pi}}\left[\frac{\cos s\pi}{s^2} - \frac{1}{s^2}\right]$$

13. State the Fourier transform of the derivative of a function (or)

If $F\left[f(x)\right] = F(s)$ **then Prove that** $F\left[x^n f(x)\right] = (-i)^n \frac{d^n}{ds^n}F(s)$

Solution:

$$F(s] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

Diferentiating this both sides with respect to s in n times

$$\frac{d^n}{ds^n}F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x).(ix)^n e^{isx}dx=(i)^n\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{x^n f(x)\}e^{isx}dx$$

$$(-i)^n\frac{d^n}{ds^n}F(s)=F\{x^n f(x)\}, Hence\ proved.$$

14. State and prove change of scale property of FT.

Solution:

Statement: If $F\left[f(x)\right]=F(s), then\, F\{f(ax)\}=\frac{1}{|a|}F(\frac{s}{a})$

Proof:

$$F\{f(ax)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{isx}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\left(\frac{s}{a}\right)t}\frac{dt}{a} \qquad [put\ ax=t$$

$$adx=dt]$$

$for\ a>0,$

$$F\{f(ax)\}=\frac{1}{a}F\left(\frac{s}{a}\right)\&$$

$for\ a<0,$

$$F\{f(ax)\}=-\frac{1}{a}F\left(\frac{s}{a}\right)$$

$$\therefore\ F\{f(ax)\}=\frac{1}{|a|}F\left(\frac{s}{a}\right), Hence\ proved.$$

15. If $F\left[f(x)\right]=F(s)$ **then Prove that** $F\left[x^2 f(x)\right]=-\frac{d^2}{ds^2}F(s)$

Solution:

$$F(s]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

Diferentiating both sides with respect to s in 2 times

$$\frac{d^2}{ds^2}F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x).(ix)^2 e^{isx}dx=(i)^2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{x^2 f(x)\}e^{isx}dx=-F[x^2 f(x)$$

$$\therefore F\{x^2 f(x)\}=-\frac{d^2}{ds^2}F(s), Hence\ proved.$$

16. Prove that $F\left[f(ax)\right]=\frac{1}{a}F\left(\frac{s}{a}\right)$ **, a>0.**

Solution:

$$F\{f(ax)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{isx}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\left(\frac{s}{a}\right)t}\frac{dt}{a} \qquad [put\ ax=t$$

$$adx=dt]$$

$$=\frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\left(\frac{s}{a}\right)t}dt$$

$$\therefore F\{f(ax)\}=\frac{1}{a}F\left(\frac{s}{a}\right), Hence\ proved.$$

17. If $F\left[f(x)\right]=F(s)$ **then Prove that** $F\left[e^{ias}f(x)\right]=F(s+a)$

Solution:

$$F\left[e^{ias}f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(s+a)}f(x)dx=F(s+a)\ \ [\because F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}f(x)dx]$$

18. Prove $F_c[f''(x)] = -s^2 F_c(s) - \sqrt{\frac{2}{\pi}} f'(0).$

Solution:

$$F_c[f''(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f''(x)\cos sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\, d[f'(x)]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left[\cos sx . f'(x)\right]_0^\infty - \int_0^\infty f'(x)(-\sin sx).s dx\right] = \sqrt{\frac{2}{\pi}}[-f'(0)] + \sqrt{\frac{2}{\pi}} s\int_0^\infty f'(x)\sin sx dx$$

$$= -\sqrt{\frac{2}{\pi}}[f'(0)] + \sqrt{\frac{2}{\pi}} s\int_0^\infty \sin sx\, d[f(x)]dx = -\sqrt{\frac{2}{\pi}}[f'(0)] + s\sqrt{\frac{2}{\pi}}\left[\left[\sin sx f(x)\right]_0^\infty - \int_0^\infty f(x)\cos sx.s dx\right]$$

$$= -\sqrt{\frac{2}{\pi}}[f'(0)] - s^2\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx = -\sqrt{\frac{2}{\pi}}[f'(0)] - s^2 F_c(s), \textit{Hence proved}$$

19. Find complex Fourier transform of $f(x) = \begin{cases} x, & for\ |x| \le a \\ 0, & for\ |x| > a \end{cases}$

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{isx} x dx = \frac{1}{\sqrt{2\pi}}\left[\frac{xe^{isx}}{is} - \frac{e^{isx}}{(is)^2}\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{ae^{isa}}{is} + \frac{ae^{-isa}}{is} + \frac{1}{s^2}(e^{isa} - e^{-isa})\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-2ai}{s}\cos sa + \frac{2i\sin sa}{s^2}\right] = \frac{2i}{s^2} \times \frac{1}{\sqrt{2\pi}}[\sin sa - as\cos sa]$$

$$= \frac{i}{s^2} \times \sqrt{\frac{2}{\pi}}[\sin sa - as\cos sa]$$

20. Prove $F[f(x-a)] = e^{ias} F(s)$.

Solution:

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x-a)dx \quad \left[\because F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)dx\right]$$

$put\ x - a = t \qquad x \to -\infty, t \to -\infty$

$dx = dt \qquad x \to \infty, t \to \infty$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{is(t+a)} f(t)dt = \frac{e^{ias}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}dt$$

$$= \frac{e^{ias}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx = e^{isa}F(s)$$

21. Define Fourier Cosine transform and its inverse.

Solution:

FCT of f(x) is defined as

$$F_c[f(x)]\ or\ F_c[s] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx$$

Its inverse is defined by

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c[f(x)]\cos sx ds$$

22. Define Fourier Sine transform and its inverse.

Solution:

FCT of f(x) is defined as

$$F_s[f(x)]\ or\ F_s[s] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx dx$$

Its inverse is defined by

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s[f(x)]\sin sx ds$$

23. When a function is called self reciprocal with respect to FT.Give an example.

Solution:

If $F\left[f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx=F(s), then$ f(x) is said to be self reciprocal. For example $e^{\frac{-x^2}{2}}$ is self reciprocal

since $F\left[e^{\frac{-x^2}{2}}\right]=e^{\frac{-s^2}{2}}$.

24. State Parseval's Identity function of Fourier transforms.or Energy theorem function of FT.

Solution:

If F(s) is the Fourier transform of f(x), then

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx=\int_{-\infty}^{\infty}|F(s)|^2\,ds$$

25. Prove that $F_s\left[f(x)\cos ax\right]=\frac{1}{2}\left[F_s(s+a)+F_s(s-a)\right]$.

Solution:

$$F_s\left[f(x)\cos ax\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\cos ax\,dx=\frac{1}{2}\times\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\left[\sin(s+a)+\sin(s-a)\right]dx$$

$$=\frac{1}{2}\times\left[\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\left[\sin(s+a)\right]dx\right]+\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\left[\sin(s-a)\right]dx\right]\right]$$

$$=\frac{1}{2}\left[F_s(s+a)+F_s(s-a)\right]$$

26. Prove that $F_c\left[f(x)\sin ax\right]=\frac{1}{2}\left[F_s(a+s)+F_s(a-s)\right]$

Solution:

$$F_c\left[f(x)\sin ax\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin ax.\cos sx\,dx=\frac{1}{2}\times\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\left[\sin(a+s)+\sin(a-s)\right]dx$$

$$=\frac{1}{2}\times\left[\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\left[\sin(a+s)\right]dx\right]+\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\left[\sin(a-s)\right]dx\right]\right]$$

$$=\frac{1}{2}\left[F_s(a+s)+F_s(a-s)\right]$$

27. State Convolution Theorem for FT.

Solution:

$$F(f*g)=F(s).G(s)\ \ if\ \ F(s)=F\left[f(x)\right]\&G(s)=F\left[g(x)\right], then$$

$$(f*g)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t).g(x-t)dt$$

28. Find the Fourier transform of $e^{-|x|}$.

Solution:

$$F(s)=F[e^{-|x|}]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}e^{isx}dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{1}{s^2+1}\right]$$

29. Find the Fourier sine transform of $f(x)=e^{-ax}, a>0$ **and hence deduce that** $\int_0^{\infty}\frac{x\sin\alpha x}{1+x^2}dx=\frac{\pi}{2}e^{-a}$

Solution:

$$F_s\left[e^{-ax}\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\sin sx dx=\sqrt{\frac{2}{\pi}\frac{s}{a^2+s^2}}, a>0$$

By the inversion formula for sine transform

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(s)\sin sx dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)\sin sxds$$

$$\int_0^{\infty}\frac{s\sin sx}{a^2+s^2}dx=\frac{\pi}{2}\ f(x)=\frac{\pi}{2}e^{-ax}, a>0$$

$$\Rightarrow \quad \int_0^\infty \frac{s\sin \alpha s}{a^2+s^2}\,ds = \frac{\pi}{2}e^{-\alpha a}$$

Put s = x and a = 1, we get

$$\int_0^\infty \frac{x\sin mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-a}$$

PART B

1. Find the Fourier Transform of $f(x)=\begin{cases}1-x^2 & in|x|\le 1\\ 0 & in|x|\ge 1\end{cases}$. Hence prove that $\int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos\frac{s}{2}\,ds = \frac{3\pi}{16}$.
2. Show that the F .T of $f(x)=\begin{cases}a^2-x^2 & ;|x|<a\\ 0 & ;|x|>a>0\end{cases}$ is $2\sqrt{\frac{2}{\pi}}\left(\frac{\sin(as)-a.s\cos(as)}{s^3}\right)$. Hence deduce that $\int_0^\infty \frac{\sin x - x\cos x}{x^3}dx = \frac{\pi}{4}$ and $\int_0^\infty\left(\frac{\sin x - x\cos x}{x^3}\right)^2 dx = \frac{\pi}{15}$.
3. Show that the F.T of $f(x)=\begin{cases}a-|x| & ;if\ |x|<a\\ 0 & ;if\ |x|>a>0\end{cases}$ is $\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos(as)}{s^2}\right)$.hence show that (i) $\int_0^\infty \frac{\sin^4 t}{t^4}dt = \frac{\pi}{3}$. (ii) $\int_0^\infty\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$
4. Find the Fourier transform of $f(x)=\begin{cases}1-|x|, & |x|<1\\ 0, & |x|>1\end{cases}$ and hence find the value of $\int_0^\infty\left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$ and $\int_0^\infty\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$.
5. Find Fourier Transform of $f(x)$ if $f(x)=\begin{cases}1, & |x|<a\\ 0, & |x|>a>0\end{cases}$. Deduce that $\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}$ and $\int_0^\infty\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$.
6. Find Fourier Transform of $f(x)$ if $f(x)=\begin{cases}1, & |x|<2\\ 0, & |x|>2\end{cases}$ and hence evaluate $\int_0^\infty \frac{\sin x}{x}dx$ and $\int_0^\infty\left(\frac{\sin x}{x}\right)^2 dx$
7. Find the (complex) F.T of $f(x)=\begin{cases}e^{ikx} & :a<x<b\\ 0 & x<a,\ x>b\end{cases}$
8. Find the Fourier Transforms of $e^{-a|x|}$, $a>0$.Hence deduce that (i) $F(xe^{-a|x|}) = i\sqrt{\frac{2}{\pi}}\frac{2as}{\left(a^2+s^2\right)^2}$ (ii) $\int_0^\infty \frac{dx}{\left(x^2+a^2\right)^2} = \frac{\pi}{4a^3}$ (iii) $\int_0^\infty \frac{\cos xt}{a^2+t^2}dt = \frac{\pi}{2a}e^{-a|x|}$.
9. Find the Fourier transform of $f(x)=\frac{1}{\sqrt{|x|}}$.
10. Find the infinite Fourier transform of the function $e^{-a^2x^2}$.Hence deduce the infinite Fourier transform of $e^{\frac{-x^2}{2}}$.
11. Find F.S.T and F.C.T of e^{-ax} and hence evaluate $(i)\int_0^\infty \frac{a^2\,dx}{\left(x^2+a^2\right)^2}$ (ii) $\int_0^\infty \frac{x^2dx}{\left(x^2+a^2\right)^2}$

12. Obtain the F.C.T of $1-x^2$ *in* (0,1) and hence deduce that $\int_0^\infty \frac{x\cos x-\sin x}{x^3}\cos\left(\frac{x}{2}\right)dx=\frac{3\pi}{16}$.

13. Find the Fourier sine and cosine transform of $\frac{e^{-ax}-e^{-bx}}{x}$.

14. Find the Fourier cosine transform of e^{-x^2}.

15. Find the Fourier sine transform of the function $f(x)=\frac{e^{-ax}}{x}$.

16. Find the infinite Fourier sine transform of the function $f(x)=\frac{e^{-ax}}{x}$. hence deduce the infinite Fourier sine transform of $\frac{1}{x}$.

17. Find the Fourier sine transform of $\frac{1}{x}$.

18. (i) Prove that $e^{\frac{-x^2}{2}}$ is self reciprocal under Fourier transform.

(ii) Prove that $f(x)=xe^{\frac{-x^2}{2}}$ is self reciprocal with respect to FCT.

19. Find Fourier sine and cosine transform of e^{-ax}, $a>0$ and deduce $\int_0^\infty \frac{s\sin sx\,ds}{s^2+a^2}$ and $\int_0^\infty \frac{\cos sx\,ds}{s^2+a^2}$

20. Find the Fourier Cosine Transform of $f(x)=e^{-a^2x^2}$ and hence find the Fourier Cosine Transform of $e^{\frac{-x^2}{2}}$ and Fourier Sine Transform of $xe^{\frac{-x^2}{2}}$.

21. (i) State and prove convolution theorem in Fourier transform
(ii) State and Prove Parseval's Identity in Fourier transform.

22. Using Parseval's identity calculate $\int_0^\infty \frac{x^2}{(a^2+x^2)^2}dx \text{ if } a>0$.

23. Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using Fourier transformation

24. Evaluate $\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}$ using Fourier transformation.

25. Solve for f(x) from the integral equation

$$\int_0^\infty f(x)\sin sx\,dx=\begin{cases}1 & ;0\le s<1\\ 0 & ;1<s>2\\ 0 & ;s\ge 2\end{cases}$$

26. Find the function whose Fourier sine transform is $\frac{e^{-as}}{s}$, $(a>0)$

27. Solve the integral equation $\int_0^\infty f(x)\cos\lambda x\,dx=e^{-\lambda}$, where λ =0.

Unit V - Z - Transforms
PART - A

1. Find $Z\left[\frac{a^n}{n!}\right]$ in Z – transform.

Soln: $Z\left[\frac{a^n}{n!}\right]=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{a}{z}\right)^n=1+\frac{1}{1!}\left(\frac{a}{z}\right)+\frac{1}{2!}\left(\frac{a}{z}\right)^2+\ldots=e^{a/z}$

2. Find $Z\left[e^{-iat}\right]$ using Z- Transform.

Soln:

$$Z\left[e^{-iat}\right] = Z[\cos at] - i\, z[\sin at] = \frac{z(z-\cos aT)}{z^2-2z\cos aT+1} - i\frac{z\sin aT}{z^2-2z\cos aT+1}$$

3. State and Prove initial value theorem in Z – Transform.

Soln: Statement: *If* $Z[f(n)] = f(z)$, *then* $f(0) = \lim_{z\to\infty} F(z)$

Proof: $F[z] = \sum_{n=0}^{\infty} f(n)\, z^{-n} = f(0) + \frac{f(1)}{Z} + \frac{f(2)}{Z^2} + ...$

$$\therefore \lim_{z\to\infty} F(z) = f(0).$$

4. Find the Z- transform of (n+1) (n+2).

Soln: $z[(n+1)(n+2)] = z[n^2+3n+2] = z(n^2) + 3z(n) + 2z(1)$

$$= \frac{z^2+z}{(z-1)^3} + \frac{3z}{(z-1)^2} + \frac{2z}{z-1}$$

$$= \frac{z^2+z+3z^2-3z+2z(z^2-2z+1)}{(z-1)^3}$$

$$z[(n+1)(n+2)] = \frac{2z^3}{(z-1)^3}.$$

5. Find the Z - Transform of (n+2).

Soln:

$$Z[n+2] = Z(n) + z(2) = \frac{z}{(z-1)^2} + \frac{2z}{z-1} = \frac{2z(z-1)+z}{(z-1)^2} = \frac{2z^2-z}{(z-1)^2}.$$

6. State the final value theorem in Z – transforms.

Soln: *If* $[f(n)] = F[z]$ *then* $\lim_{n\to\infty} f(n) = \lim_{Z\to 1}(z-1)\, F(z)$.

7. Find $Z\left[\frac{1}{n}\right]$.

Soln:

$$Z\left[\frac{1}{n}\right] = \sum_{n=0}^{\infty}\frac{1}{n} z^{-n} = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + ... = -\log\left(1-\frac{1}{z}\right) = \log\left(\frac{z}{z-1}\right)$$

8. Express $z[f(n+1)]$ **in terms of** $\bar{f}(z)$.

Soln: $Z\{f(n+1)\} =$

$$\sum_{n=0}^{\infty} f(n+1)z^{-n} = z\sum_{n=0}^{\infty} f(n+1)z^{-(n+1)} = z\left[\sum_{n=0}^{\infty} f(n)z^{-n} - f(0)\right].$$

$$= z\left[\overline{f(z)} - f(0)\right]$$

9. Find the values of $z[f(n)]$ **when** $f(n) = n\,a^n$.

Soln: W.K.T $z[n] = \frac{z}{(z-1)^2}$

$$\therefore z[n\,a^n] = \left[\frac{z}{(z-1)^2}\right]_{z\to z/a} = \frac{z/a}{\left(z/a - 1\right)^2} = \frac{az}{(z-a)^2}.$$

10. If $\mathbf{Z}[f(n)] = F(z)$ **then prove that** $\mathbf{Z}[a^n f(n)] = \mathbf{Z}[f(n)]_{Z\to \frac{Z}{a}}$.

Soln: $z[a^n f(n)] = \sum_{n=0}^{\infty} z^{-n} a^n f(n) = \sum_{n=0}^{\infty} f(n)\left(\frac{a}{z}\right)^n = F\left(z/a\right)$

11. Find $\mathbf{Z}\left[\sin\frac{n\pi}{2}\right]$.

Soln: $z[\sin n\theta] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$, *put* $\theta = \frac{\pi}{2}$

$$\therefore z\left[\sin\tfrac{n\pi}{2}\right] = \frac{z}{z^2+1} \quad \left(\because \sin \pi/2 = 1\right).$$

12. Find $Z[a^n]$

Soln: $z[a^n] = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty}\frac{a^n}{z^n} = \sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n$

Put $\frac{a}{z} = x$, $\therefore \sum_{n=0}^{\infty} x^n \Rightarrow 1 + x + x^2 + ... = (1-x)^{-1} = \left(1-\frac{a}{z}\right)^{-1} = \left(\frac{z}{z-a}\right).$

13. State Convolution theorem on Z- transforms.

Soln: *If* $z[f(n)] = F[z]$ *and* $z[g(n)] = G[z]$, then $z[f(n)*g(n)] = F[z]\,G[z]$, where $f(n)*g(n)$ is defined

$$f*g = \sum_{k=0}^{\infty} f(n)\,g(n-k).$$

14. Find $z[n^2]$

Soln: $z[n^2] = z[n.n]$

$$\Rightarrow \therefore z[n.n] = -z.F'(z) = -z\,\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$\left\{\because w.k.t\ z(n) = \frac{z}{(z-1)^2}\right\} = \frac{-z(z-1)(z+2z)}{(z-1)^4} = \frac{z(z+1)}{(z-1)^3}.$$

15. State and Prove first shifting theorem.

Soln: Statement:

If $z\,f(n) = F(z)$ *,then* $z\left[e^{-an}\ f(n)\right] = F(z\ e^{a})$

Proof: By definition, $z[f(n)] = \sum_{n=0}^{\infty} f(n)\ z^{-n} = F(z)$ ……… (1)

$$\therefore z\left[e^{-an}\ f(n)\right] = \sum_{n=0}^{\infty} e^{-an}\ f(n).\ z^{-n} = \sum_{n=0}^{\infty} f(n)\left(e^{-an}.\ z^{-n}\right) = \sum_{n=0}^{\infty} f(n)\ z^{-n}\,(e^{a})^{-n}$$

$$= \sum_{n=0}^{\infty} f(n)\ (ze^{a})^{-n} = F(z\,e^{a})\ by\ (1).$$

16. Define inverse Z- transform

Soln: *If* $z[f(n)] = F(z)$ *,then* $z^{-1}[F(z)] = f(n),$ *where* $f(n)$ *is* called the inverse Z- transform of $F(z)$

17. Find the Z- transform of $\dfrac{1}{(n+1)!}$

Soln:

$$z\left[\frac{1}{(n+1)!}\right] = z\sum_{n=0}^{\infty}\frac{1}{(n+1)!}z^{-(n+1)} = z\left[\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\ldots\right] = z\left(e^{1/z}-1\right)$$

.

18. State and Prove second shifting theorem.

Soln:

Statement

: *If* $z[f(t)] = F(z)$ *then* $z[f(t+T)] = z[f(z) - f(0)]$

Proof:

$$z[f(t+T)] = \sum_{n=0}^{\infty} f(nT+T)\,z^{-n} = \sum_{n=0}^{\infty} f[(n+1)T]\,z^{-n}$$

$$= z\sum_{n=0}^{\infty} f[(n+1)T]\,z^{-(n+1)}$$

$$= \therefore z\sum_{k=1}^{\infty} f[kT]\,z^{-k}$$

Put k = n+1

$$z[f(t+T)] = z\left[\sum_{k=0}^{\infty} f[kT]z^{-k} - f(0)\right] = z[f(z) - f(0)].$$

19. Find the Z- transform of $2^n * n$

Soln:

$$z[2^n * n] = z[2^n].z[n] = \frac{z}{z-2}.\frac{z}{(z-1)^2} = \frac{z^2}{(z-1)^2(z-2)}$$

20. Find the Z- transform of $\cos n\theta$ *and* $\sin n\theta$**.**

Soln: $z\left[e^{-in\theta}\right]=z\left[(e^{-i\theta})^{n}\right]=\dfrac{z}{z-e^{-i\theta}}$

$$=\frac{z(z-e^{i\theta})}{(z-e^{-i\theta})(z-e^{i\theta})}=\frac{z\left(z-\cos\theta\right)-iz\sin\theta}{z^{2}-z\left(e^{-i\theta}+e^{i\theta}\right)+1}$$

$$\therefore \quad z\left[\cos n\theta-i\sin n\theta\right]=\frac{z\left(z-\cos\theta\right)-iz\cos}{z^{2}-2z\cos\theta+1}.$$

21. Find the Z – transform of $\dfrac{a^{n}}{n!}e^{-a}$

Soln:

$$z\left[e^{-a}.\frac{a^{n}}{n!}\right]=\sum_{n=0}^{\infty}e^{-a}\frac{a^{n}}{n!}z^{-n}=e^{-a}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{a}{z}\right)^{n}=e^{-a}.e^{a/z}=e^{a\left(1/z-1\right)}$$

.

22. Find $z(n.3^{n})$

Soln: w. k .t $z\left(3^{n}\right)=\dfrac{z}{z-3}$

$$\therefore z(n.3^{n})=-z.\frac{d}{dz}\left(\frac{z}{z-3}\right)=\frac{3z}{\left(z-3\right)^{2}}$$

23. Using convolution theorem evaluate $z^{-1}\left[\dfrac{z^{2}}{(z-1)(z-3)}\right]$

Soln: w. k. t $z^{-1}\left[\dfrac{z}{(z-1)}\right]=1^{n}$ & $z^{-1}\left[\dfrac{z}{(z-3)}\right]=3^{n}$

Now

$$z^{-1}\left[\frac{z^{2}}{(z-1)(z-3)}\right]=z^{-1}\left[\frac{z}{(z-1)}.\frac{z}{(z-3)}\right]=1^{n}*3^{n}$$

$$=\sum_{k=0}^{n}1^{k}.3^{n-k}=3^{n}+3^{n-1}+3^{n-2}+\ldots+3+1$$

$$=1+3+\ldots+3^{n}==\frac{3^{n+1}-1}{3-1}.$$

24. Evaluate $z^{-1}\left[\dfrac{z}{z^{2}+7z+10}\right]$

Soln:

Let $F[z]=\dfrac{z}{z^{2}+7z+10}$ *ie.,* $\dfrac{F[z]}{z}=\dfrac{1}{z^{2}+7z+10}=\dfrac{1}{(z+2)(z+5)}$

$$=\frac{1}{3}\left[\frac{1}{(z+2)}-\frac{1}{(z+5)}\right] \quad \textit{by Partial fraction}$$

$$\therefore F[z]=\frac{1}{3}\frac{z}{(z+2)}-\frac{1}{3}\frac{z}{(z+5)}$$

$$\Rightarrow z^{-1}\left[F[z]\right]=\frac{1}{3}(-2)^{n}-\frac{1}{3}(-5)^{n}.$$

25. Solve $y_{n+1}-2y_{n}=0$ **given** $y_{0}=2$**.**

Taking Z-transform $(z-2)\ z(y(n))=2z$

$\Rightarrow Z[y(n)]=\dfrac{2z}{z-2}$ $\quad\therefore\ y(n)=z^{-1}\left[\dfrac{2z}{z-2}\right]=2.2^{n}=2^{n+1}$.

26. **Using convolution theorem, find** $Z^{-1}\left[\dfrac{z^2}{(z+a)^2}\right]$**.**

$$Z^{-1}\left[\frac{z^2}{(z+a)^2}\right]=$$

$$Z^{-1}\left[\frac{z}{z+a}.\frac{z}{z+a}\right]=Z^{-1}\left[\frac{z}{z+a}\right]*Z^{-1}\left[\frac{z}{z+a}\right]=(-a)^n*(-a)^n$$

$$=\sum_{k=0}^{\infty}(-a)^k(-a)^{n-k}=(n+1)(-a)^n\,.$$

PART B

1. Find $\mathbf{Z}$-transform of (i) $\mathbf{Z}\left[\dfrac{1}{(n+1)}\right]$ (ii) $\mathbf{Z}\left[\dfrac{2n+3}{(n+1)(n+2)}\right]$

   (iii) $Z\left[\dfrac{1}{n(n+1)}\right]$ (iv) $Z\left[\cos\dfrac{n\pi}{2}\right]$ (v) $\mathbf{Z}[n(n-1)]$ (vi) $\mathbf{Z}$ [$r^n\cos n\theta$]

   (vii) $\mathbf{Z}$ [$e^{-at}\cos bt$] (viii) Z[n^3] (ix) Z[$e^{-t}t^2$]
2. State and prove final value theorem in $\mathbf{Z}$-transform.
3. State and prove second shifting theorem in $\mathbf{Z}$-transform.
4. State and prove Convolution theorem on Z-transform.
5. Prove that (i) $Z(n)=\dfrac{z}{(z-1)^2},|z|>1$

   (ii) $Z\left(\dfrac{1}{n}\right)=\log\left(\dfrac{z}{z-1}\right)$ *if* $|z|>1,\ n>0.$

TYPE-I (Using convolution theorem)

(i) $\mathbf{Z}^{-1}\left[\dfrac{z^2}{(z-1)(z-3)}\right]$ (ii) $\mathbf{Z}^{-1}\left[\dfrac{z^2}{(z-a)^2}\right]$

(iii) $\mathbf{Z}^{-1}\left[\dfrac{8z^2}{(2z-1)(4z-1)}\right]$ (iv) $Z^{-1}\left[\dfrac{z^2}{(z-a)(z-b)}\right]$

(v) $Z^{-1}\left[\dfrac{8z^2}{(2z-1)(4z+1)}\right]$

TYPE-II (Using partial fraction)

(i) $\mathbf{Z}^{-1}\left[\dfrac{z}{z^2+11z+24}\right]$ (ii) $\mathbf{Z}^{-1}\left[\dfrac{8z^2}{(2z-1)(4z+1)}\right]$

(iii) $\mathbf{Z}^{-1}\left[\dfrac{z(z^2-z+2)}{(z-1)^2(z+1)}\right]$ (iv) $\mathbf{Z}^{-1}\left[\dfrac{z^3}{(z-1)^2(z-2)}\right]$

(v) $\mathbf{Z}^{-1}\left[\dfrac{z^2}{(z+2)(z^2+4)}\right]$ (vi) $Z^{-1}\left[\dfrac{10z}{(z-1)(z-2)}\right]$

(vii) $Z^{-1}\left[\dfrac{z}{z^2+7z+10}\right]$ (viii) $\left[\dfrac{3z^2+z}{z^3-3z^2+4}\right]$

(ix) $Z^{-1}\left[\dfrac{4z^3}{(2z-1)^2(z-1)}\right]$

TYPE-III (Using residue theorem)

(i) $\mathbf{Z}^{-1}\left[\dfrac{z(z+1)}{(z-1)^3}\right]$ (ii) $\mathbf{Z}^{-1}\left[\dfrac{z^2}{(z+2)(z^2+4)}\right]$ (iii) $Z^{-1}\left[\dfrac{10z}{(z-1)(z-2)}\right]$

(iv) $Z^{-1}\left[\dfrac{z^2-3z}{(z-5)(z+2)}\right]$ (v) $Z^{-1}\left[\dfrac{z}{z^2-2z+2}\right]$

(vi) $Z^{-1}\left[\dfrac{9z^3}{(3z-1)^2(z-2)}\right]$

4. **Solving difference equation:**

(a) $y(k+2)-4y(k+1)+4y(k)=0;\ n\ge 0;\quad y(0)=1;\ y(1)=0$

(b) $y(n)-4y(n-1)+4y(n-2)=0;\ n\ge 2;\quad y(0)=3;\ y(1)=-2$

(c) $y_{n+2}+6y_{n+1}+9y_n=2^n;\ n\ge 0;\quad y_0=0;\ y_1=0$

(d) $y_{n+2}-5y_{n+1}+6y_n=4^n;\ n\ge 0;\quad y_0=0;\ y_1=1$

(e) $y(n+3)-3y(n+1)+2y(n)=0;\ n\ge 0;\quad y(0)=1;\ y(1)=0;\ y(2)=8$

(f) $y_{n+2}+y_n=n2^n;\ n\ge 0;\quad y_0=a;\ y_1=b$

(g) $y(n+2)-3y(n+1)+2y(n)=2^n$, given $y(0)=y(1)=0$.

(h) $y(n+2)+3y(n+1)+2y(n)=0$, *given* $y(0)=1, y(1)=2$.

(i) $y(n+2)-3y(n+1)-10y(n)=0$, given $y(0)=1, y(1)=0$.

(j) $Y_{n+2}+2Y_{n+1}+Y_n=n$ given $Y_0=Y_1=0$

(k) $u_{n+2}+4u_{n+1}+3u_n=3^n$ with $u_0=0, u_1=1$

(l) $y_{n+2}-7y_{n+1}+12y_n=2^n$. Given $y_0=y_1=0$

5. Form the difference equation from

(i) $Y(n)=(A+Bn)2^n$ (ii) $Y_n=A3^n$ (iii) $Y(n)=a+b3^n$

(iv) $Y(n)=(an+bn^2)$

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